Optimal control under uncertainty: Application to the issue of CAT bonds

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Abstract

We propose a general framework for studying optimal issue of CAT bonds in the presence of uncertainty on the parameters. In particular, the intensity of arrival of natural disasters is inhomogeneous and may depend on unknown parameters. Given a prior on the distribution of the unknown parameters, we explain how it should evolve according to the classical Bayes rule. Taking these progressive prior-adjustments into account, we characterize the optimal policy through a quasi-variational parabolic equation, which can be solved numerically. We provide examples of application in the context of hurricanes in Florida.

1 Introduction

We consider an insurer or a reinsurer who holds a portfolio in non-life insurance exposed to one or several natural disasters. He can issue one or several CAT bonds[1] in order to reduce the risk taken, see e.g. [7] or [8] for a general introduction to CAT bonds.

The first CAT bonds where issued at the end of the 1990s and the market is globally increasing, with a total risk capital outstanding greater that USD 30 billion at the end of 2017, see [1] and [5]. CAT bonds give a strong alternative to the classical reinsurance market.

However, issuing a CAT bond leads to the choice of several parameters, as the layer e.g. and the date of issuance. The coupon is not a priori perfectly known as well as the claim distribution. Moreover, the global warming will lead to an increase of several natural disasters which is a source of uncertainty on the distribution of future claims. For example, in [11], the authors estimate that if the temperature rises of 2.5 degrees in the next decades, the frequency of Hurricanes in North Atlantic will rise by 30%.

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[1]Catastrophe bonds, or CAT bonds, are tradable floating rate notes. The risk associated with a CAT bond is not linked to the default of one entity (state or corporate) but is related to the occurrence of a catastrophe.

The aim of this paper is to provide a rigorous continuous-time framework in which we can establish the optimal behavior policy in issuing CAT bonds, taking into account the uncertainty described above as the risk evolution.

The coupon of the CAT bond is generally not known in advance, even its distribution is not always clearly fixed. We therefore need to model it as a random variable whose distribution depends on unknown parameters. It is the same for the distribution of the natural disasters.

The particular case of acting on a system with partially unknown response distributions has been studied in [3] in a Brownian framework, see the references therein for the case of discrete settings. They fix a prior distribution on the unknown parameter and introduce a stochastic process on the space of measures which leads to a dynamic programming principle and a PDE characterization of the value function (in the viscosity solution sense).

In this paper, the natural disasters will be represented by a random Poisson measure[2] and two parameters are unknown: the distribution of the severity of the natural disasters and the intensity of their arrivals. As in [3], we allow the agent to issue new CAT bonds at any time, the actions are discrete but chosen in a continuous time framework.

To the best of our knowledge, the study of such a general problem with an application to the CAT bonds seems to be new in the literature, even in the case where all parameters are known. From a mathematical point of view, the main difficulty comes from the fact that the conditional distribution on the unknown parameters evolves continuously and jumps at the occurrence times of a catastrophic event. In [3], it was only evolving when an action was taken on the system. For tractability, we assume that the associated process remains in a finite-dimensional space which can be linked smoothly to a subset of $\mathbb{R}^d$ for some $d \geq 1$.

Although the model presented below has been designed for the particular case of CAT bonds, it is quite general from a mathematical view-point and can be applied to all cases where the agent faces a random Poisson measure and can issue contracts from which he pays a premium and receives a specific payoff depending on some event.

2 The framework

2.1 General framework

All over this paper, $D([0,T],\mathbb{R}^d)$ is the Skorohod space of càdlàg[3] functions from $[0,T]$ into $\mathbb{R}^d$, $\mathbb{P}$ is a probability measure on this space, and $T > 0$ is a fixed time horizon.

We consider three Polish spaces: $(U^\lambda, \mathcal{B}(U^\lambda))$, $(U^\gamma, \mathcal{B}(U^\gamma))$ and $(U^\upsilon, \mathcal{B}(U^\upsilon))$ that will support three unknown parameters, respectively λ_0, γ_0 and υ_0. Here $\mathcal{B}(.)$ denotes the Borel σ-algebra. We set $U := (U^\lambda, U^\gamma, U^\upsilon)$.

Let $N(dt, du)$ be a random Poisson measure with compensator $\nu(dt, du)$ such that ν is finite on $(\mathbb{R}^{d*}, \mathcal{B}(\mathbb{R}^{d*}))$ where $\mathbb{R}^{d*} := \mathbb{R}^d \setminus \{0_{\mathbb{R}^d}\}$. The intensity of the random Poisson measure is supposed to be inhomogeneous of intensity $s \mapsto \Lambda(s, \lambda_0)$ where λ_0 is a random variable

[2]The activity of the random Poisson measure will be finite, by construction

[3]continue à droite, limite à gauche (Right continuous with left limits)

valued in U^λ. The jump distribution is assumed to be $\Upsilon(\gamma_0, \cdot)$ where γ_0 is a random variable valued in U^γ. We denote by $\mathbf{M}^\lambda$ a subset of the set of Borel probability measures on U^λ and by $\mathbf{M}^\gamma \otimes \mathbf{M}^\upsilon =: \mathbf{M}$ the product of two locally compact subsets of the set of Borel probability measures, respectively on U^γ and U^υ, endowed with the weak topology.

We also allow an additional randomness when acting on the system and consider another Polish space $(E, \mathcal{B}(E))$ on which is defined a family $(\epsilon_i)_{i\geq 1}$ of i.i.d. random variables with common probability measure $\mathbb{P}_\epsilon$ on $\mathcal{B}(E)$.

On the product space $\Omega := D([0,T], \mathbb{R}^d) \times U \times E^{\mathbb{N}^*}$, we consider the family of measures $\{\mathbb{P} \times \overline{m} \times \mathbb{P}_\epsilon^{\otimes \mathbb{N}^*}, \overline{m} \in \overline{\mathbf{M}}\}$ where $\overline{\mathbf{M}} := \mathbf{M}^\lambda \otimes \mathbf{M}$. We denote by $\mathbb{P}_{\overline{m}}$ an element of this family whenever $\overline{m} \in \overline{\mathbf{M}}$ is fixed. The operator $\mathbb{E}_{\overline{m}}$ is the expectation associated with $\mathbb{P}_{\overline{m}}$. Note that $N(dt, du)$ and $(\epsilon_i)_{i\geq 1}$ are independent under each $\mathbb{P}_{\overline{m}}$. For $\overline{m} \in \overline{\mathbf{M}}$ given, we let $\mathbb{F}^{\overline{m}} := (\mathcal{F}_t^{\overline{m}})_{t\geq 0}$ denote the $\mathbb{P}_{\overline{m}}$-augmentation of the filtration $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ defined by $\mathcal{F}_t := \sigma(N([0,s] \times \cdot)_{s\leq t}, \lambda_0, \gamma_0, \upsilon_0, (\epsilon_i)_{i\geq 1})$. Hereafter, all random variables are considered with respect to the probability space $(\Omega, \mathcal{F}_T^{\overline{m}}, \mathbb{P}_{\overline{m}})$ with $\overline{m} \in \overline{\mathbf{M}}$ given by the context.

2.2 CAT Bond framework

In this framework, $d \in \mathbb{N}^*$ is the number of perils. The insurer has some exposure related to these perils and may issue CAT bonds to reduce the risk taken. The random Poisson measure represents the arrival of claims. The intensity of arrival is $s \mapsto \Lambda(s, \lambda_0)$ in which λ_0, valued in U^λ, may be unknown to the insurer. The dependence in time may represent the seasonality or a structural change, for example caused by the global warming.

The measure $m^\lambda \in \mathbf{M}^\lambda$ is the initial knowledge of the insurer on λ_0 and will evolve through the observations of N, whose jumps model the arrival of natural disasters. The severity distribution of the claims may also be unknown, it depends on the unknown parameter γ_0, valued in U^γ. An initial prior is given as an element $m^\gamma \in \mathbf{M}^\gamma$. Acting on the system consists in issuing a CAT bond, which means transferring a part of the risk to the market. The equilibrium premium that the insurer will pay is random (since it comes from the law of supply and demand and is not know when the decision to issue is taken), and the distribution may not be perfectly known. We assume that it depends on the unknown parameter υ_0, valued in U^υ. Its prior distribution is represented by some $m^\upsilon \in \mathbf{M}^\upsilon$.

We fix a maximum of $n \in \mathbb{N}$ possible CAT bonds in term of risk covered. The possible risk coverages are denoted by $(A_j)_{1\leq j\leq n}$ with $A_j \subset \mathcal{B}(\mathbb{R}^{d*})$ in which $\mathcal{B}(\mathbb{R}^{d*})$ denotes all Borel sets of $\mathbb{R}^{d*}$. In practice, it will represent the layer of one peril for one region, and then, if for $j = 1$, it is the first dimension (risk) of N which is covered, A_j will have the form $[a, +\infty[\times \mathbb{R} \times \ldots \times \mathbb{R}$ with $a > 0$. If a claim $u \in \mathbb{R}^{d*}$ satisfies $u \in A_1$, it will give a payoff of the form $(u_1 - a)$ bounded by some $b > 0$ associated with this layer (the layer is $[a, a + b]$).

2.3 The controlled system

Let $\mathbf{A} \subset \mathbb{R}^{d+1}$ be a non-empty compact set. Let $\ell \in \mathbb{R}_+^*$ be the time-length of each action on the controlled system. Given $\overline{m} \in \overline{\mathbf{M}}$, we denote by $\Phi^{\circ,\overline{m}}$ the collection of random variables $\phi = (\tau_i^\phi, \alpha_i^\phi)_{i\geq 1}$ on $(\Omega, \mathcal{F}_T^{\overline{m}})$ with values in $\mathbb{R}_+ \times \mathbf{A}$ such that $(\tau_i^\phi)_{i\geq 1}$ is a non-decreasing sequence of $\mathbb{F}^{\overline{m}}$-stopping times and each α_i is $\mathcal{F}^{\overline{m}}_{\tau_i}$-measurable for $i \geq 1$. We shall write $\alpha_i^\phi := (k_i^\phi, n_i^\phi) \in \mathbf{A}$ where k_i^ϕ and n_i^ϕ are $\mathbb{R}^d$ and $\mathbb{R}$-valued. To each k_i^ϕ, we associate a non-empty closed set $A_{k_i^\phi} \subset \mathbb{R}^{d*}$ through a one-to-one map.

The τ_i^ϕ's will be the times at which a CAT bond is issued. The fixed value ℓ is the time-length (or maturity) of all CAT bonds. In $\alpha_i^\phi := (k_i^\phi, n_i^\phi) \in \mathbf{A}$, n_i^ϕ is related to the notional and $A_{k_i^\phi}$ is the layer chosen for one peril and one region: it is the characteristics of the CAT bonds associated with the risk covered. If a natural disaster occurs and its severity is in the layer $A_{k_i^\phi}$, i.e. the random Poisson measure has a jump in $A_{k_i^\phi}$, then the CAT bonds ends and the reinsurer gains a payoff proportional to the notional n_i^ϕ.

We denote by ϑ_i^ϕ the end of the i-th CAT Bond defined by:

$$\vartheta_i^\phi := \inf\{t > \tau_i^\phi, N(\{t\} \times A^\phi_{k_i}) = 1\} \wedge (\tau_i^\phi + \ell). \tag{2.1}$$

Remark 2.1. *According to the definition of $(\vartheta_i^\phi)_{i\geq 1}$, it can happen that $\vartheta_{i_1}^\phi = \vartheta_{i_2}^\phi$ for $i_1 \neq i_2$. Moreover,*

$$\tau_i^\phi < \vartheta_i^\phi \leq \tau_i^\phi + \ell.$$

We are now in position to describe the controlled state process. Given some initial data $(t,x) \in [0,T] \times \mathbb{R}^d$, and $\phi \in \Phi^{\circ,\overline{m}}$, we let $X^{t,x,\phi}$ be a strong solution on $[t,T]$ of

$$\begin{aligned} X := x &+ \int_t^{\cdot} \mu(s, X_s)ds + \int_t^{\cdot} \int_{\mathbb{R}^d} \beta(s, X_{s-}, u) N(ds, du) \\ &+ \sum_{i\geq 1} \mathbf{1}_{\{t \leq \tau_i^\phi < \cdot\}} H(\tau_i^\phi, X_{\tau_i^\phi}, \alpha_i^\phi) \\ &+ \sum_{i\geq 1} \mathbf{1}_{\{t\vee\tau_i^\phi \leq \cdot\}} \int_{t\vee\tau_i^\phi}^{\cdot\wedge\vartheta_i^\phi} \overline{C}(s, r_i^\phi) ds \\ &+ \sum_{i\geq 1} \mathbf{1}_{\{t \leq \vartheta_i^\phi \leq \cdot\}} F(\vartheta_i^\phi, X_{\vartheta_i^\phi -}, X_{\tau_i^\phi}, r_i^\phi, \alpha_i^\phi, \vartheta_i^\phi - \tau_i^\phi, u_i) \mathbf{1}_{\{\vartheta_i^\phi - \tau_i^\phi \neq \ell\}}, \end{aligned} \tag{2.2}$$

in which $r_i^\phi := \mathfrak{C}_0(\tau_i^\phi, X_{\tau_i^\phi -}, \alpha_i^\phi, \upsilon, \epsilon_i)$ with $\mathfrak{C}_0 : [0,T] \times \mathbb{R}^d \times \mathbf{A} \times U^\upsilon \times E \to \mathbb{R}$ a measurable function and u_i is the jump size of the random Poisson measure N at ϑ_i.

To guarantee existence and uniqueness of the above, we make the following standard assumptions.

Assumption 2.1. $\mu : [0,T] \times \mathbb{R}^d \mapsto \mathbb{R}^d$, $\beta : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \mapsto \mathbb{M}^d$ *and* $\overline{C} : [0,T] \times \mathbb{R} \mapsto \mathbb{R}$, *are assumed to be measurable, continuous, and Lipschitz with linear growth in their second argument, uniformly in the other ones.*

The maps $H : [0,T] \times \mathbb{R}^d \times \mathbf{A}$, and $F : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \times \mathbf{A} \times [0,\ell] \times \mathbb{R}^d$ are assumed to be measurable. Moreover, H (resp. F) has linear growth in its second (resp. third) component.

This dynamics means the following. Without any CAT bond, the process X follows a pure jump process with a drift described by the first line of (2.2). The second line refers to a jump of the whole process when a CAT bond is issued, for example, with a fixed cost. The third line represents the instantaneous cash flows generated by the closed and current active CAT bonds. The last line represents the final cash flow if the policy ends before the maturity.

The first component of the process X will be the cash. The second may record the aversion of the market for the CAT bonds: when a natural disaster occurs, it jumps and then decrease again over time. The function μ can be the drift associated with some interest rate or to the decrease of the risk aversion of the market when no natural disaster occurs. The function β can represent the exposure in cash of the reinsurer for each peril, and also the sensitivity of the CAT bond market (for the second component of X) when a natural disaster occurs. The function H represents an initial cost to issue the CAT bond. The function $\bar{C}$ is the continuous premium paid by the reinsurer for the CAT bond and r_i^ϕ is the level of the coupon (a random variable which is determined by an unknown parameter υ and a noise ϵ_i). The function F is the payout, $\vartheta_i^\phi - \tau_i^\phi \neq \ell$ means that the CAT Bond ends with an event.

We denote by $\mathbb{F}^{t,x,\overline{m},\phi} := \left(\mathcal{F}_s^{t,x,\overline{m},\phi}\right)_{s\geq 0}$ the $\mathbb{P}_{\overline{m}}$-augmentation of the filtration generated by $\left(X^{t,x,\phi}, \sum_{i\geq 1} r_i^\phi \mathbf{1}_{[\tau_i^\phi,+\infty[}, N([t,s]\times\cdot)_{s\geq t}\right)$.

For $\kappa \in \mathbb{N}^*$, we say that $\phi \in \Phi^{\circ,\overline{m}}$ belongs to $\Phi_\kappa^{t,x,\overline{m}}$ if the condition

$$\sum_{i\geq 1} \mathbf{1}_{\{\tau_i^\phi < t \leq \vartheta_i^\phi\}} \leq \kappa \quad \forall t \leq T \tag{2.3}$$

holds. The set $\Phi_\kappa^{t,x,\overline{m}}$ is the set of admissible controls. The constraint (2.3) refers to the fact that the controller cannot have more than κ simultaneous running CAT bonds at each time.

Note that $X^{t,x,\phi}$ has a jump of size $H(\tau_i^\phi, X_{\tau_i^\phi}^{t,x,\phi}, \alpha_i^\phi)$ at each τ_i^ϕ and is left-continuous at this point, whereas it is right-continuous at each ϑ_i. This allows to observe a jump from the left from the random Poisson measure and then issue immediately a new CAT bond, leading to an immediate jump of X from the right. The process $X^{t,x,\phi}$ defined above is làdlàg.

2.4 The CAT bonds process

We need to keep track of how many CAT bonds are running, and which parameters are associated with. Corresponding to the definition of the process X in (2.2), the effect of a CAT bond will be measured by the value of $(X_{\tau_i^\phi}^{t,x,\phi}, r_i^\phi, \alpha_i^\phi)$ determined at τ_i, for $(t,x) \in [0,T]\times\mathbb{R}^d, \phi \in \Phi_\kappa^{t,x,\phi}$. Moreover, a CAT bond will end from a jump or after the time-length ℓ. We need to define a process which will keep track of this information. We introduce the sets $\mathbf{C} := \left((\mathbb{R}^d \times \mathbb{R} \times \mathbf{A}) \cup \partial\right)^\kappa$, $\mathbf{L} := ([0,\ell[\cup\partial)^\kappa$, in which

- An element of the set $\mathbb{R}^d \times \mathbb{R} \times \mathbf{A}$ represents the initial parameters of the CAT bond;
- An element of the set $[0,\ell[$ represents the time-length elapsed of a running CAT bond;
- The point ∂ represents the absence of CAT bond, it is a cemetery point.

The set of CAT bonds is

$$\mathbf{CL} := \{(c,l) \in \mathbf{C} \times \mathbf{L} \ \mid \ c_j = \partial \Longleftrightarrow l_j = \partial, \ \forall\ 1 \leq j \leq \kappa\}$$

and we denote by $\overline{\mathbf{CL}}$ its closure. We set $\mathbf{K} := \{0, \ldots, \kappa\}$ and we define by $\mathcal{P}(\mathbf{K})$ the set of subsets of $\mathbf{K}$. We can now define the sets $\mathbf{CL_J}$ with $\mathbf{J} \in \mathcal{P}(\mathbf{K})$:

$$\mathbf{CL_J} := \{(c,l) \in \mathbf{CL} \ \mid \ j \in \mathbf{J} \Longleftrightarrow c_j \neq \partial, \ \forall\ 1 \leq j \leq \kappa\}$$

which represent the sets of CAT Bonds in which there is CAT Bonds running exactly in the indexes of $\mathbf{J}$.

Moreover, for $(c,l) \in \mathbf{CL} \backslash \mathbf{CL_K}$, we introduce:

$$\Pi^0(c,l) := \min\{1 \leq j \leq \kappa : c^j = \partial\},$$

which is the first index with no CAT bond.

For $z := (t,x,c,l) \in \mathbf{Z} := [0,T] \times \mathbb{R}^d \times \mathbf{CL}$ and a control $\phi \in \Phi_\kappa^{t,x,\overline{m}}$, we now define the process $((C,L)_s^{z,\phi,j})_{t \leq s \leq T}^{1 \leq j \leq \kappa}$ valued in $\mathbf{CL}$ and denoted hereafter (C,L) for ease of notation. The process (C,L) will jump at the $\tau_i's$ (new CAT bond) and at the ϑ_i's (end of one or several CAT bonds). C will be a pure jump process whereas the indexes of L will evolve continuously over time, recall that it represents the elapsed time-length of the CAT bonds.

We now define the functions associated with the jumps of (C,L). The first one, denoted by $\mathfrak{C}_+$, represents the arrival of one new CAT bond with parameters $(x,r,a) \in \mathbb{R}^d \times \mathbb{R} \times \mathbf{A}$ and is defined by

$$\begin{aligned} \mathfrak{C}_+ : (\mathbf{CL} \backslash \mathbf{CL_K}) \times \mathbb{R}^d \times \mathbb{R} \times \mathbf{A} &\to \mathbf{CL} \\ (c,l;x,r,a) &\mapsto \mathfrak{C}_+(c,l;x,r,a) \end{aligned}$$

where, if we write (c_+, l_+) for $\mathfrak{C}_+(c,l;x,r,a)$,

$$\begin{aligned} &(c_+, l_+)_{\Pi^0(c,l)} := ((x,r,a),0), \\ &(c_+, l_+)_j = (c,l)_j \qquad\qquad j \neq \Pi^0(c,l). \end{aligned} \tag{2.4}$$

The second function, denoted by $\mathfrak{C}_-$, represents the end of the CAT bonds by an event associated with the random Poisson measure, of severity $u \in \mathbb{R}^{d*}$, and is defined by

$$\begin{aligned} \mathfrak{C}_- : \mathbf{CL} \times \mathbb{R}^{d*} &\to \mathbf{CL} \\ (c,l;u) &\mapsto \mathfrak{C}_-(c,l;u). \end{aligned}$$

Nonetheless, several CAT bonds may end with a single event. We define the set of indexes in $c \in \mathbf{C}$ which end after the natural disaster $u \in \mathbb{R}^{d*}$, by

$$\mathcal{J}(c;u) := \left\{ j \in \{1, \ldots, \kappa\} \mid c_j \neq \partial, u \in A_{k_j} \right\}. \tag{2.5}$$

Using this set, $\mathfrak{C}_-(c,l;u)$ is defined simply through its j-component

$$\mathfrak{C}_-(c,l;u)_j := \begin{cases} \partial \times \partial & \text{if } j \in \mathcal{J}(c;u) \\ (c,l)_j & \text{if } j \notin \mathcal{J}(c;u) \end{cases}, \ 1 \leq j \leq \kappa\,. \tag{2.6}$$

It remains to consider the case where a CAT Bond ends because $l_j = \ell$ for some $1 \le j \le \kappa$. We define:

$$\mathfrak{C}^{\ell}_{-} : (\overline{\mathbf{CL}} \backslash \overline{\mathbf{CL}}_{\emptyset}) \to \mathbf{CL}$$
$$(c, l) \mapsto \mathfrak{C}^{\ell}_{-}(c, l),$$

where, for all $1 \le j \le \kappa$,

$$\mathfrak{C}^{\ell}_{-}(c, l)_j = (\partial \times \partial)\mathbf{1}_{\{l_j = \ell\}} + (c, l)_j \mathbf{1}_{\{l_j \neq \ell\}}.$$

We are now in position to define the processes $C^{z,\phi}$ and $L^{z,\phi}$ for $\phi \in \Phi^{t,x,\overline{m}}_{\kappa}$. The process evolves at τ^{ϕ}_i and ϑ^{ϕ}_i, for $i \ge 1$, according to:

$$\begin{aligned} (C, L)^{z,\phi}_{\tau^{\phi}_i +} &:= \mathfrak{C}_{+}((C, L)^{z,\phi}_{\tau^{\phi}_i}); X^{z,\phi}_{\tau^{\phi}_i}, r^{\phi}_i, \alpha^{\phi}_i); \\ (C, L)^{z,\phi}_{\vartheta^{\phi}_i} &:= \mathbf{1}_{\{\vartheta^{\phi}_i < \tau^{\phi}_i + \ell\}} \mathfrak{C}_{-}((C, L)^{z,\phi}_{\vartheta^{\phi}_i -}, u_i) + \mathbf{1}_{\{\vartheta^{\phi}_i = \tau^{\phi}_i + \ell\}} \mathfrak{C}^{\ell}_{-}((C, L)^{z,\phi}_{\vartheta^{\phi}_i -}). \end{aligned} \tag{2.7}$$

Elsewhere, $C^{z,\phi}$ is constant. For $1 \le j \le \kappa$, $L^{z,\phi,j}$ evolves according to:

$$dL^{z,\phi,j}_t = \mathbf{1}_{\{L^{z,\phi,j}_t \neq \partial\}} dt.$$

This closes the definition of the process (C, L). Note that we separated both the initial parameters with the elapsed time-length since the second one will play a different role in the PDE characterization in consequence of its continuous part.

Remark 2.2. *If $c \mapsto \Pi(c) := \#\{j \in \mathbf{K} : c^j \neq \partial\}$, the process $C^{z,\phi}$ (and also, by construction, $L^{z,\phi}$) satisfies :*

$$\begin{aligned} &\Pi(C^{z,\phi}_s) \le \kappa, \ \forall s \in [t, T], \ \mathbb{P}_{\overline{m}} - a.s. \\ &\Pi(C^{z,\phi}_{\tau^{\phi}_i}) \le \kappa - 1, \ \forall i \ge 1, \ \mathbb{P}_{\overline{m}} - a.s.. \end{aligned}$$

We also give a metric on $\overline{\mathbf{CL}}$.

Definition 2.1. *We associate to $\overline{\mathbf{CL}}$ the metric $\mathfrak{d}$ defined by*

$$\begin{aligned} \mathfrak{d}\left[(c, l), (c', l')\right] := &\sum_{j \in \mathbf{J} \cap \mathbf{J}'} \left[\|c_j - c'_j\|^2 + (l_j - l'_j)^2\right] + \sum_{j \in \mathbf{J} \backslash \mathbf{J}'} (\|c_j\|^2 + l_j^2) \\ &+ \sum_{j \in \mathbf{J}' \backslash \mathbf{J}} (\|c'_j\|^2 + (l'_j)^2) + Card(\mathbf{J} \Delta \mathbf{J}'), \end{aligned}$$

where $\mathbf{J}$ and $\mathbf{J}'$ are respectively the set of running CAT bonds of parameters (c, l) and (c', l').

Remark 2.3. *For $z := (t, x, c, l) \in \mathbf{Z}$, we shall write $X^{z,\phi}$ for the process X starting with the CAT bonds (c, l) and $\mathbb{F}^{z,\phi}$ the same filtration as $\mathbb{F}^{t,x,\overline{m},\phi}$ but also starting with the CAT bonds (c, l). Note that (C, L) is adapted $\mathbb{F}^{z,\overline{m},\phi}$-adapted. Moreover, we define $\Phi^{z,\overline{m}}_{\kappa}$ as $\Phi^{t,x,\overline{m}}_{\kappa}$ but, again, starting with CAT bonds (c, l).*

2.5 Bayesian updates

Obviously, the prior $\overline{m} \in \overline{\mathbf{M}}$ will evolve over time. Recall that $\overline{\mathbf{M}} := \mathbf{M}^\lambda \otimes \mathbf{M}$ and denote by $\overline{m} := (m^\lambda, m^\gamma, m^\upsilon)$ the corresponding element. The observation of X over time will lead to a continuous update of m^λ, whereas m^γ will be updated by observing the size of a jump from N and the measure m^υ will be updated by acting on the system at times τ_i^ϕ. This leads to the definition of the process $M := (M^\lambda, M^\gamma, M^\upsilon)$ valued in $\overline{\mathbf{M}}$. We first focus on m^λ.

2.5.1 Evolution of the intensity

We start with the assumption associated with the unknown and inhomogeneous intensity of the random Poisson measure.

Assumption 2.2. *For all $m^\lambda \in \mathbf{M}^\lambda$,*

i) $\int_s^t \Lambda(u, \lambda_0)du < +\infty$ *$m^\lambda - a.s.$, for all* $0 \le s \le t$.

ii) $t \mapsto \Lambda(t, \lambda_0)$ *is a càdlàg process* $m^\lambda - a.s.$

iii) For almost every $s \ge 0$ *such that* $\Lambda(s, \lambda_0) > 0$ $m^\lambda - a.s.$*, there exists* $h_0 > 0$ *and* $K > 0$ *such that* $\int_s^{s+h} \Lambda(u, \lambda_0)du \le Kh\Lambda(s, \lambda_0)$ *for all* $h \le h_0$.

iv) $\int_0^{+\infty} \Lambda(u, \lambda_0)du = +\infty$ $m^\lambda - a.s.$

Between two jumps of the random Poisson measure, the probability measure associated with λ_0 will evolve continuously. When a jump occurs, it jumps as well. We first deal with what happens between two jumps.

Remark 2.4. *Remark that, since a càdlàg function has at most a countable set of points of discontinuity, under ii) of Assumption 2.2 we have*

$$\int_s^t \Lambda(u, \lambda_0)e^{-\int_\alpha^u \Lambda(v,\lambda_0)dv}du = e^{-\int_\alpha^s \Lambda(v,\lambda_0)dv} - e^{-\int_\alpha^t \Lambda(v,\lambda_0)dv} \quad m^\lambda\text{-}a.e. \tag{2.8}$$

for almost all $0 \le \alpha \le s \le t$.

Given $B \in \mathcal{B}(U^\lambda)$, we set $M_s^{t,m^\lambda}(B) := \mathbb{E}_{\overline{m}}\left(\mathbf{1}_{\{\lambda_0 \in B\}} | \mathcal{F}_s^{z,\overline{m},\phi}\right)$ for $z = (t, x, c, l)$ and $\phi \in \Phi_\kappa^{z,\overline{m}}$. We shall see below that M_s^{t,m^λ} does not depend on x and ϕ. From now on, we denote by $(\zeta_i)_{i \ge 1}$ the jump times associated with the random Poisson measure.

Lemma 2.1. *For all* $z = (t, x, c, l) \in \mathbf{Z}$ *and* $s > t$,

$$M_s^{t,m^\lambda}(B)\mathbf{1}_{\{\zeta_i \le s < \zeta_{i+1}\}} = \mathfrak{M}_\lambda(B; \zeta_i, s)\mathbf{1}_{\{\zeta_i \le s < \zeta_{i+1}\}}$$

where

$$\mathfrak{M}_\lambda(B; \zeta_i, s) := \frac{\int_B e^{-\int_{\zeta_i}^s \Lambda(u,\lambda)du} M_{\zeta_i}^{t,m^\lambda}(d\lambda)}{\int_{\mathbb{R}_+} e^{-\int_{\zeta_i}^s \Lambda(u,\lambda)du} M_{\zeta_i}^{t,m^\lambda}(d\lambda)}\mathbf{1}_{\{\zeta_i \le s\}}.$$

Proof. Let φ be a Borel bounded function on $D([0,T],\mathbb{R}^{d+1})$. Set $\xi^\phi := \sum_{i\geq 1} r_i^\phi \mathbf{1}_{[\tau_i^\phi,+\infty[}$, $\delta X^i := X^{z,\phi}_{\cdot\vee\zeta_i} - X^{z,\phi}_{\zeta_i}$, and $\delta\xi^i := \xi_{\cdot\vee\zeta_i} - \xi_{\zeta_i}$. Note that $\delta\xi^i_{\cdot\wedge s}\mathbf{1}_{\{\zeta_i\leq s<\zeta_{i+1}\}}$ is $\sigma(\mathcal{F}^{z,m,\phi}_{\zeta_i}\cup\sigma(\upsilon,(\epsilon_j)_{1\leq j\leq K}))$-measurable. We can find a Borel measurable map $\overline{\varphi}$ such that

$$\varphi(X^{z,\phi}_{\cdot\wedge s},\xi^\phi_{\cdot\wedge s})\mathbf{1}_{\{\zeta_i\leq s<\zeta_{i+1}\}} = \overline{\varphi}(X^{z,\phi}_{\cdot\wedge\zeta_i},\xi^\phi_{\cdot\wedge\zeta_i},\delta X^i_{\cdot\wedge s},\delta\xi^i_{\cdot\wedge s})\mathbf{1}_{\{\zeta_i\leq s<\zeta_{i+1}\}}.$$

In view of Remark 2.4, it then follows:

$$\begin{aligned}
&\mathbb{E}_{\overline{m}}\left(\mathbf{1}_{\{\lambda_0\in B\}}\mathbf{1}_{\{\zeta_i\leq s<\zeta_{i+1}\}}\varphi(X^{z,\phi}_{\cdot\wedge s},\xi^\phi_{\cdot\wedge s})\right)\\
&= \mathbb{E}_{\overline{m}}\left(\mathbf{1}_{\{\lambda_0\in B\}}\mathbf{1}_{\{\zeta_i\leq s<\zeta_{i+1}\}}\overline{\varphi}(X^{z,\phi}_{\cdot\wedge\zeta_i},\xi^\phi_{\cdot\wedge\zeta_i},\delta X^i_{\cdot\wedge s},\delta\xi^i_{\cdot\wedge s})\right)\\
&= \mathbb{E}_{\overline{m}}\left(\int_{\mathbb{R}_+}\mathbf{1}_{\{\lambda_0\in B\}}\mathbf{1}_{\{\zeta_i\leq s<u\}}\overline{\varphi}(X^{z,\phi}_{\cdot\wedge\zeta_i},\xi^\phi_{\cdot\wedge\zeta_i},\delta X^i_{\cdot\wedge s},\delta\xi^i_{\cdot\wedge s})\Lambda(u,\lambda_0)e^{-\int_{\zeta_i}^u\Lambda(v,\lambda_0)dv}du\right)\\
&= \mathbb{E}_{\overline{m}}\left(\mathbf{1}_{\{\lambda_0\in B\}}\overline{\varphi}(X^{z,\phi}_{\cdot\wedge\zeta_i},\xi^\phi_{\cdot\wedge\zeta_i},\delta X^i_{\cdot\wedge s},\delta\xi^i_{\cdot\wedge s})\int_{\mathbb{R}_+}\mathbf{1}_{\{\zeta_i\leq s<u\}}\Lambda(u,\lambda_0)e^{-\int_{\zeta_i}^u\Lambda(v,\lambda_0)dv}du\right)\\
&= \mathbb{E}_{\overline{m}}\left(\mathbf{1}_{\{\lambda_0\in B\}}\overline{\varphi}(X^{z,\phi}_{\cdot\wedge\zeta_i},\xi^\phi_{\cdot\wedge\zeta_i},\delta X^i_{\cdot\wedge s},\delta\xi^i_{\cdot\wedge s})\mathbf{1}_{\{\zeta_i\leq s\}}e^{-\int_{\zeta_i}^s\Lambda(v,\lambda_0)dv}\right)\\
&= \mathbb{E}_{\overline{m}}\left(\overline{\varphi}(X^{z,\phi}_{\cdot\wedge\zeta_i},\xi^\phi_{\cdot\wedge\zeta_i},\delta X^i_{\cdot\wedge s},\delta\xi^i_{\cdot\wedge s})\mathbf{1}_{\{\zeta_i\leq s\}}\int_B e^{-\int_{\zeta_i}^s\Lambda(v,\lambda)dv}M^{t,m^\lambda}_{\zeta_i}(d\lambda)\right)\\
&= \mathbb{E}_{\overline{m}}\left(\overline{\varphi}(X^{z,\phi}_{\cdot\wedge\zeta_i},\xi^\phi_{\cdot\wedge\zeta_i},\delta X^i_{\cdot\wedge s},\delta\xi^i_{\cdot\wedge s})\mathbf{1}_{\{\zeta_i\leq s\}}\mathfrak{M}_\lambda(B;\zeta_i,s)\int_{\mathbb{R}_+}e^{-\int_{\zeta_i}^s\Lambda(v,\lambda)dv}M^{t,m^\lambda}_{\zeta_i}(d\lambda)\right)\\
&= \mathbb{E}_{\overline{m}}\left(\overline{\varphi}(X^{z,\phi}_{\cdot\wedge\zeta_i},\xi^\phi_{\cdot\wedge\zeta_i},\delta X^i_{\cdot\wedge s},\delta\xi^i_{\cdot\wedge s})\mathbf{1}_{\{\zeta_i\leq s<\zeta_{i+1}\}}\mathfrak{M}_\lambda(B;\zeta_i,s)\right)\\
&= \mathbb{E}_{\overline{m}}\left(\varphi(X^{z,\phi}_{\cdot\wedge s},\xi^\phi_{\cdot\wedge s})\mathbf{1}_{\{\zeta_i\leq s<\zeta_{i+1}\}}\mathfrak{M}_\lambda(B;\zeta_i,s)\right)
\end{aligned}$$

This shows that $M^{t,m^\lambda}_s(B)\mathbf{1}_{\{\zeta_i\leq s<\zeta_{i+1}\}} = \mathfrak{M}_\lambda(B;\zeta_i,s)\mathbf{1}_{\{\zeta_i\leq s<\zeta_{i+1}\}}$ $\mathbb{P}_{\overline{m}}$-a.s. □

Lemma 2.2. *For all $m^\lambda\in\mathbf{M}^\lambda$ and almost all $s\geq t$, we have*

i)

$$\int_{U^\lambda}\Lambda(s,\lambda)M^{t,m^\lambda}_s(d\lambda) < +\infty \quad \mathbb{P}_{\overline{m}} - a.s.$$

ii)

$$\int_{U^\lambda}\Lambda(\zeta_i,\lambda)M^{t,m^\lambda}_{\zeta_i-}(d\lambda) < +\infty \quad \mathbb{P}_{\overline{m}} - a.s, \quad i\geq 1.$$

iii)

$$\int_{U^\lambda}\Lambda(s,\lambda)M^{t,m^\lambda}_{s-}(d\lambda) < +\infty \quad \mathbb{P}_{\overline{m}} - a.s.$$

Proof. **Step 1**. For almost all $\lambda\in U^\lambda$, we fix $\mathcal{N}_\lambda\subset[0,T]$ the set of discontinuity of $t\mapsto\Lambda(t,\lambda)$ which is, at most, countable. We introduce:

$$\mathcal{N}^c := \{\forall i\geq 1, \zeta_i\notin\mathcal{N}_{\lambda_0}\}.$$

We shall show that $\mathbb{P}(\mathcal{N}^c) = 1$ by showing that $\mathbb{P}(\zeta_i \in \mathcal{N}_{\lambda_0}) = 0$ for all $i \geq 1$. Fix $i \geq 1$ and remark that, given $\lambda \in U^\lambda$, the distribution of $\zeta_i \mid \{\lambda_0 = \lambda\}$ is absolutely continuous with respect to the Lebesgue measure. Denote by $f_{i|\lambda}$ a corresponding density function. Then,

$$\mathbb{P}_{\overline{m}}(\zeta_i \in \mathcal{N}_{\lambda_0}) = \int_{U^\lambda} \left[\int_{\mathbb{R}_+} \mathbf{1}_{\mathcal{N}_\lambda}(z) f_{i|\lambda}(z) dz \right] dm^\lambda(\lambda) = \int_{U^\lambda} 0 \, dm^\lambda(\lambda) = 0.$$

Step 2. We show *i)*. We set:

$$K_i(s) := \left(\int_{U^\lambda} e^{-\int_{\zeta_i}^s \Lambda(u,\lambda) du} M_{\zeta_i}^{t,m^\lambda}(d\lambda) \right)^{-1} \leq K_i(\zeta_{i+1}) \quad \text{on } \{\zeta_i \leq s < \zeta_{i+1}.$$

We have, by *i)* of Assumption 2.2,

$$K_i(\zeta_{i+1}) < +\infty.$$

Moreover, by Fubini's Lemma and Remark 2.4,

$$\begin{aligned}\int_{\zeta_i}^{\zeta_{i+1}} \int_{U^\lambda} \Lambda(s,\lambda) e^{-\int_{\zeta_i}^s \Lambda(u,\lambda) du} M_{\zeta_i}^{t,m^\lambda}(d\lambda) ds &= \int_{U^\lambda} \int_{\zeta_i}^{\zeta_{i+1}} \Lambda(s,\lambda) e^{-\int_{\zeta_i}^s \Lambda(u,\lambda) du} ds M_{\zeta_i}^{t,m^\lambda}(d\lambda) \\ &= \int_{U^\lambda} [1 - e^{-\int_{\zeta_i}^{\zeta_{i+1}} \Lambda(u,\lambda) du}] M_{\zeta_i}^{t,m^\lambda}(d\lambda) < +\infty,\end{aligned}$$

on $\mathcal{N}^c$. On the other hand, using Lemma 2.1,

$$\int_{\zeta_i}^{\zeta_{i+1}} \int_{U^\lambda} \Lambda(s,\lambda) M_s^{t,m^\lambda}(d\lambda) ds \leq K_i(\zeta_{i+1}) \int_{\zeta_i}^{\zeta_{i+1}} \int_{U^\lambda} \Lambda(s,\lambda) e^{-\int_{\zeta_i}^s \Lambda(u,\lambda) du} M_{\zeta_i}^{t,m^\lambda}(d\lambda) ds < +\infty$$

on $\mathcal{N}^c$. This shows that, for almost all $s \geq t$,

$$\mathbf{1}_{\{\zeta_i < s < \zeta_{i+1}\}} \int_{U^\lambda} \Lambda(s,\lambda) M_s^{t,m^\lambda}(d\lambda) < +\infty \text{ on } \mathcal{N}^c.$$

This leads to the result since $\zeta_i \to +\infty$ when $i \to +\infty$ for almost all ω.

Step 3. We show *ii)*. Since M^{t,m^λ} evolves continuously on all $]\zeta_i, \zeta_{i+1}[$, we also have,

$$\int_{U^\lambda} \Lambda(\zeta_i -, \lambda) M_{\zeta_i -}^{t,m^\lambda}(d\lambda) < +\infty \ \ \mathbb{P}_{\overline{m}} - a.s.$$

Moreover, on $\mathcal{N}^c$, ζ_i cannot be on a discontinuity of Λ by construction, $i \geq 1$. Then, we have, on $\mathcal{N}^c$,

$$\int_{U^\lambda} \Lambda(\zeta_i, \lambda) M_{\zeta_i -}^{t,m^\lambda}(d\lambda) < +\infty.$$

Step 4. We show *iii)*. We introduce:

$$A := \{s \in [t,T] : m^\lambda [\Lambda(s,\lambda_0) = 0] < 1\}.$$

Recall that, by construction, $M_s^{t,m^\lambda} << m^\lambda$ for all $s \geq t$. If $s \in A$, $\int_{U^\lambda} \Lambda(s,\lambda) M_s^{t,m^\lambda}(d\lambda) = 0 < +\infty$. If $s \notin A$, the distribution of ζ_i is equivalent to the Lebesgue measure and then, by *ii)*, we get the result. $\square$

We now look at the intensity at the observation of a jump ζ_i.

Lemma 2.3. *For all $z = (t, x, c, l) \in \mathbf{Z}$ and $B \in \mathcal{B}(U^\lambda)$,*

$$M^{t,m^\lambda}_{\zeta_i}(B) = \frac{\int_B \Lambda(\zeta_i, \lambda) M^{t,m^\lambda}_{\zeta_i -}(d\lambda)}{\int_{U^\lambda} \Lambda(\zeta_i, \lambda) M^{t,m^\lambda}_{\zeta_i -}(d\lambda)}, \ i \geq 1.$$

Proof. We use the same notations as in the proof of Lemma 2.1.

1. For ease of notation, we set $B_i(\zeta) := \{\zeta_{i-1} < s, \zeta_i \in [s, s+h], s+h < \zeta_{i+1}\}$. For $s > 0$, we show that, with $\zeta_0 := 0$,

$$M^{t,m^\lambda}_{s+h}(B) \mathbf{1}_{B_i(\zeta)} = \mathfrak{M}'_\lambda(B; M^{t,m^\lambda}_{s-}, s, h) \mathbf{1}_{B_i(\zeta)}, \tag{2.9}$$

where

$$\mathfrak{M}'_\lambda(B; M^{t,m^\lambda}_{s-}, s, h) := \frac{\int_B \left[\int_s^{s+h} \Lambda(u, \lambda) du\right] e^{-\int_s^{s+h} \Lambda(u,\lambda) du} M^{t,m^\lambda}_{s-}(d\lambda)}{\int_{U^\lambda} \left[\int_s^{s+h} \Lambda(u, \lambda) du\right] e^{-\int_s^{s+h} \Lambda(u,\lambda) du} M^{t,m^\lambda}_{s-}(d\lambda)}.$$

Let φ be a Borel bounded function of $D([0,T], \mathbb{R}^{d+1})$, we can find a Borel measurable map $\overline{\varphi}$ such that

$$\varphi(X^{z,\phi}_{\cdot \wedge s+h}, \xi^{\phi}_{\cdot \wedge s+h}) \mathbf{1}_{B_i(\zeta)} = \overline{\varphi}(X^{z,\phi}_{\cdot \wedge s}, \xi^{\phi}_{\cdot \wedge s}, \delta X^i_{\cdot \wedge s+h}, \delta \xi^i_{\cdot \wedge s+h}) \mathbf{1}_{B_i(\zeta)}.$$

We shall write $\overline{\varphi}(X, \xi)$ for $\overline{\varphi}(X^{z,\phi}_{\cdot \wedge s}, \xi^{\phi}_{\cdot \wedge s}, \delta X^i_{\cdot \wedge s+h}, \delta \xi^i_{\cdot \wedge s+h})$. It then follows:

$$\begin{aligned}
&\mathbb{E}_{\overline{m}}\left(\mathbf{1}_{\{\lambda_0 \in B\}} \mathbf{1}_{B_i(\zeta)} \varphi(X^{z,\phi}_{\cdot \wedge s+h}, \xi^{\phi}_{\cdot \wedge s+h})\right) \\
&= \mathbb{E}_{\overline{m}}\left(\mathbf{1}_{\{\lambda_0 \in B\}} \mathbf{1}_{B_i(\zeta)} \overline{\varphi}(X, \xi)\right) \\
&= \mathbb{E}_{\overline{m}}\left(\int_{U^\lambda} \mathbf{1}_{\{\lambda \in B\}} \mathbf{1}_{\{\zeta_{i-1} < s\}} \overline{\varphi}(X, \xi) \left[\int_s^{s+h} \Lambda(u, \lambda) du\right] e^{-\int_s^{s+h} \Lambda(u,\lambda) du} M^{t,m^\lambda}_{s-}(d\lambda)\right) \\
&= \mathbb{E}_{\overline{m}}\left(\overline{\varphi}(X, \xi) \mathbf{1}_{\{\zeta_{i-1} < s\}} \int_B \left[\int_s^{s+h} \Lambda(u, \lambda) du\right] e^{-\int_s^{s+h} \Lambda(u,\lambda) du} M^{t,m^\lambda}_{s-}(d\lambda)\right) \\
&= \mathbb{E}_{\overline{m}}\left(\overline{\varphi}(X, \xi) \mathbf{1}_{\{\zeta_{i-1} < s\}} \mathfrak{M}'_\lambda(B; M^{t,m^\lambda}_{s-}, s, h) \int_{U^\lambda} \left[\int_s^{s+h} \Lambda(u, \lambda) du\right] e^{-\int_s^{s+h} \Lambda(u,\lambda) du} M^{t,m^\lambda}_{s-}(d\lambda)\right) \\
&= \mathbb{E}_{\overline{m}}\left(\overline{\varphi}(X, \xi) \mathbf{1}_{B_i(\zeta)} \mathfrak{M}'_\lambda(B; M^{t,m^\lambda}_{s-}, s, h)\right) \\
&= \mathbb{E}_{\overline{m}}\left(\varphi(X^{z,\phi}_{\cdot \wedge s+h}, \xi^{\phi}_{\cdot \wedge s+h}) \mathbf{1}_{B_i(\zeta)} \mathfrak{M}'_\lambda(B; M^{t,m^\lambda}_{s-}, s, h)\right)
\end{aligned}$$

This shows that (2.9) hold $\mathbb{P}_{\overline{m}}$-a.s.

2. For $i = 1$, on $\{\zeta_1 \geq s\}$, by Lemma 2.2, $\Lambda(s, \lambda_0) \in L^1(M^{t,m^\lambda}_{s-})$ for almost all s. Using *iii)* of Assumption 2.2, by the dominated convergence theorem, we deduce that

$$M^{t,m^\lambda}_s(B) \mathbf{1}_{\{\zeta_0 < s, \zeta_1 = s\}} = \frac{\int_B \Lambda(s, \lambda) M^{t,m^\lambda}_{s-}(d\lambda)}{\int_{U^\lambda} \Lambda(s, \lambda) M^{t,m^\lambda}_{s-}(d\lambda)},$$

i.e., since the law of ζ_1 is absolutely continuous with respect to the Lebesgue measure,

$$M_{\zeta_1}^{t,m^\lambda}(B) = \frac{\int_B \Lambda(\zeta_1,\lambda) M_{\zeta_1-}^{t,m^\lambda}(d\lambda)}{\int_{U^\lambda} \Lambda(\zeta_1,\lambda) M_{\zeta_1-}^{t,m^\lambda}(d\lambda)} \quad \mathbb{P}_{\overline{m}} \text{ - a.s.}$$

Since almost surely, $\zeta_{i+1} > \zeta_i, i \geq 1$, and since the law of each ζ_i is absolutely continuous with respect to the Lebesgue measure, we deduce the result by a straightforward induction. □

We provide a sufficient condition for Assumption 2.2 to hold.

Lemma 2.4. *Assume that Λ can be written as follows:*

$$\Lambda(s,\lambda) = \mathbf{1}_A(s) \sum_{i=1}^n f_i(s) g_i(\lambda),$$

for all $(s,\lambda) \in [0,T] \times U^\lambda$ where:

- *A is a Borel set of $[0,T]$ such that $s \mapsto \mathbf{1}_A(s)$ is càdlàg,*
- *$(g_i)_{1\leq i\leq n} : U^\lambda \mapsto \mathbb{R}_+$ are measurable and positive,*
- *$(f_i)_{1\leq i\leq n} : [0,T] \mapsto \mathbb{R}_+$ are càdlàg, positive and locally bounded by below.*

Then Assumption 2.2 holds.

Proof. Let $\epsilon > 0$. Since for each $1 \leq i \leq p$, f_i is right continuous and locally bounded by bellow, there exists $h_0^i > 0$ and $c_i > 0$ such that, for all $0 \leq h \leq h_0^i$, $c_i \leq f_i(s+h) \leq f_i(s) + \epsilon$. Let $h_0 := \min_{1\leq i\leq p} h_0^i$ and $c := \min_{1\leq i\leq p} c^i$. Then, for $0 \leq h \leq h_0$

$$\begin{aligned}
\int_s^{s+h} \Lambda(u,\lambda) du &= \sum_{i=1}^p g_i(\lambda) \int_s^{s+h} \mathbf{1}_A(u) f_i(u) du \leq \sum_{i=1}^p g_i(\lambda) \int_s^{s+h} (f_i(s)+\epsilon) du \\
&\leq h\Lambda(s,\lambda) \left(1 + \frac{\epsilon \sum_{i=1}^p g_i(\lambda)}{\sum_{i=1}^p f_i(s) g_i(\lambda)}\right) \leq h\Lambda(s,\lambda)\left(1 + c^{-1}\epsilon\right).
\end{aligned}$$

□

2.5.2 Evolution of the parameters γ_0 and v_0

We use the notations of Section 2.5.1. We define $M_s^{t,m^\gamma}(B) := \mathbb{E}_{\overline{m}}\left(\mathbf{1}_{\{\gamma\in B\}} | \mathcal{F}_s^{z,\overline{m},\phi}\right)$ and $M_s^{z,m^v,\phi}(B) := \mathbb{E}_{\overline{m}}\left(\mathbf{1}_{\{v\in B\}} | \mathcal{F}_s^{z,\overline{m},\phi}\right)$.

Between two jumps of the random Poisson measure, no information about the size distribution of the jumps is revealed, and therefore, about γ_0. Whereas no information is revealed about v between two jumps from our control. In this case, both processes should remain constant. At the i-th Poisson jump of size u_i, the process M^{t,m^γ} should evolve according to the classical Bayes rule. The process $M^{z,m^v,\phi}$ should evolve at the time the j-th CAT bonds with the coupon c_j is issued according to, again, the Bayes rule.

Lemma 2.5. *Fix $s \geq 0$. Assume that, for almost all $\gamma \in U^\gamma$, the claim size distribution is dominated by some common measure $\mu_\circ$. We have*

$$
\begin{aligned}
M_s^{t,m^\gamma}(B)\mathbf{1}_{\{\zeta_i \leq s < \zeta_{i+1}\}} &= M_{\zeta_i}^{t,m^\gamma}(B)\mathbf{1}_{\{\zeta_i \leq s < \zeta_{i+1}\}}\\
M_{\zeta_i}^{t,m^\gamma}(B) &= \mathfrak{M}_\gamma(M_{\zeta_i -}^{t,m^\gamma}(B); U_i)
\end{aligned}
$$

in which

$$
\mathfrak{M}_\gamma(m_\circ^\gamma; u_\circ) = \frac{\int_B \mathrm{q}_\gamma(u_\circ \mid \gamma) dm_\circ^\gamma(\gamma)}{\int_U \mathrm{q}_\gamma(u_\circ \mid \gamma) dm_\circ^\gamma(\gamma)}.
$$

for almost all $(m_\circ^\gamma, u_\circ) \in \mathbf{M}^\gamma \times \mathbb{R}^{d*}$, in which $\mathrm{q}_\gamma(u_\circ \mid \gamma)$ is the conditional density, with respect to $m_\circ^\gamma$, of observing a jump of size $u_\circ$ knowing $\{\gamma_0 = \gamma\}$.

Moreover,

$$
\begin{aligned}
M_s^{t,m^\upsilon,\phi}(B)\mathbf{1}_{\{\tau_j \leq s < \tau_{j+1}\}} &= M_{\tau_j}^{t,m^\upsilon,\phi}(B)\mathbf{1}_{\{\tau_j \leq s < \tau_{j+1}\}}\\
M_{\tau_j}^{t,m^\upsilon,\phi}(B) &= \mathfrak{M}_\upsilon(M_{\tau_j -}^{t,m^\upsilon,\phi}(B); r_j, \tau_j, X_{\tau_j -}^{z,\phi}, \alpha_j)
\end{aligned}
$$

in which

$$
\mathfrak{M}_\upsilon(m_\circ^\upsilon; r_\circ, t_\circ, x_\circ, a_\circ) = \frac{\int_C \mathrm{q}_\upsilon(r_\circ \mid t_\circ, x_\circ, a_\circ, \upsilon) dm_\circ^\upsilon(\upsilon)}{\int_U \mathrm{q}_\upsilon(r_\circ \mid t_\circ, x_\circ, a_\circ, \upsilon) dm_\circ^\upsilon(\upsilon)}.
$$

for almost all $(m_\circ^\upsilon, r_\circ, t_\circ, x_\circ, a_\circ) \in \mathbf{M}^\upsilon \times \mathbb{R} \times [0,T] \times \mathbb{R}^d \times \mathbf{A}$, in which $\mathrm{q}_\upsilon(r_\circ \mid t_\circ, x_\circ, a_\circ, \upsilon)$ is the conditional density, with respect to $m_\circ^\upsilon$, of observing a jump of size $r_\circ$ knowing $\{\tau_j = t_\circ, X_{\tau_j -}^{z,\phi} = x_\circ, \alpha_i = a_\circ, \upsilon_0 = \upsilon\}$.

Proof. Use the same arguments as in the proof of Proposition 2.1 in [3]. □

2.6 Parametrization of the set $\mathbf{M}^\lambda$

Here, we have three measures on which will depend the value function. The one associated with the distribution of the jumps of the Poisson measure and the one from the unknown parameter evolve by a finite number of jumps on each bounded interval: the first one according to the jumps of the random Poisson process and the second one according to the impulses from the control. Those will not lead to deal with derivatives on the space of measures and a specific Itô formula nor generator of the diffusion. However, the measure associated with the parameter of the intensity evolves continuously. To deal with this, we will assume that the associated space of measures can be linked smoothly to a subset of $\mathbb{R}^k$ for some $k \geq 1$.

Assumption 2.3. *We assume that there exists an open or compact set $\mathbf{P} \subset \mathbb{R}^k$, for some $k \in \mathbb{N}^*$, and a function*

$$
\begin{aligned}
f : \mathbf{P} &\to \mathbf{M}_\lambda\\
\theta &\mapsto f(\theta),
\end{aligned}
$$

which is a homeomorphism between $\mathbf{P}$ and $\mathbf{M}_\lambda$.

Remark 2.5. *The process $P^{t,p}$ defined by:*

$$p = f^{-1}(m^\lambda), \quad P^{t,p}_s := f^{-1}(M^{t,m^\lambda}_s), \quad s \geq t,$$

remains, by construction, in $\mathbf{P}$*. Moreover, Lemma 2.1 and 2.3 provide that M^{t,m^λ} only depends on the stopping times of the jumps of the random Poisson measure on $[0,t]$, thus, M^{t,m^λ} is $\mathcal{F}^N := s \mapsto \sigma\left(N(u,\cdot), t \leq u \leq s\right)$-adapted. Then, from Assumption 2.3, $P^{t,p}$ is also $\mathcal{F}^N$-adapted. Moreover, $P^{t,p}$ does not depend on the size of the jumps.*

According to Remark 2.5, we formulate the following assumption.

Assumption 2.4. *Let $P^{t,p}$ be the process defined in Remark 2.5.*

There exists Lipschitz maps $h_1 : [0,T] \times \mathbf{P} \to \mathbb{R}^k$ and $h_2 : [0,T] \times \mathbf{P} \to \mathbb{R}^k$ with linear growth such that

$$\begin{aligned}
P^{t,p} &= p + \int_t^\cdot h_1(s, P^{t,p}_s)ds + \int_t^\cdot \int_{\mathbb{R}^{d*}} h_2(s, P^{t,p}_{s-})N(ds,du) \\
&= p + \int_t^\cdot h_1(s, P^{t,p}_s)ds + \int_t^\cdot h_2(s, P^{t,p}_{s-})dN_s,
\end{aligned}$$

where we use the notation: $dN_s := N(ds, \mathbb{R}^{d*})$.

We provide two examples in which the Assumptions 2.3 and 2.4 are fulfilled.

Example 2.1. *Assume that there exists a càdlàg function $h : [0,T] \mapsto \mathbb{R}^+$ such that $\Lambda(t,\lambda) = \lambda h(t)$ for all $t \geq 0, \lambda \in U^\lambda$. Set $m^\lambda = M^{t,m^\lambda}_t := \mathcal{G}(\alpha_t, \beta_t)$, where $\mathcal{G}$ denotes the Gamma distribution. Then, if we define*

$$(\alpha, \beta) := \left(\alpha_t + N - N_t, \beta_t + \int_t^\cdot h(u)du\right),$$

it follows that

$$M^{t,m^\lambda} = \mathcal{G}\left(\alpha, \beta\right),$$

and $P^{t,p} = (\alpha, \beta)$ satisfies Assumption 2.4.

Example 2.2. *Assume that $U^\lambda := \{\lambda_1, \ldots, \lambda_n\} \in (\mathbb{R}^*_+)^n$. Define, for $p = (p_i)_{1 \leq i \leq n}$ with $p_i > 0, 1 \leq i \leq n$, the distribution $\mathcal{D}(p)$ by:*

$$\mathcal{D}(p) := \frac{\sum_{i=1}^n p_i \delta_{\lambda_i}}{\sum_{i=1}^n p_i}.$$

Set, for $s \geq t$,

$$P^{t,p,i}_s := p^i \left[\prod_{j=N_t+1}^{N_s} \Lambda(\zeta_j, \lambda_i)\right] e^{-\int_s^t \Lambda(u,\lambda_i)du}, \quad 1 \leq i \leq n.$$

Then $M^{t,m^\lambda} = \mathcal{D}(P^{t,p})$ and the process above satisfies the stochastic differential equation:

$$P^{t,p,i} = p^i - \int_t^\cdot P^{t,p,i}_s \Lambda(s,\lambda_i)ds + \int_t^\cdot P^{t,p,i}_{s-}[\Lambda(s,\lambda_i) - 1]dN_s, \quad 1 \leq i \leq n.$$

2.7 Gain function

Given $z = (t, x, c, l) \in \mathbf{Z}$ and $(p, m) \in \mathbf{P} \times \mathbf{M}$, the aim of the controller is to maximize the expected value of the gain functional

$$\phi \in \Phi^{z,\overline{m}} \mapsto G^{z,p,m}(\phi) := g(X_T^{z,\phi}, C_T^{z,\phi}, L_T^{z,\phi}, P_T^{t,p}, M_T^{z,m,\phi}),$$

in which g is a continuous and bounded function on $\mathbb{R}^d \times \mathbf{CL} \times \mathbf{P} \times \mathbf{M}$. Recall that $C_T^{z,\phi}$ is the random variable which represents all CAT bonds which are still active at the end and that $L_T^{z,\phi}$ is the elapsed time. If there is an initial cost when a CAT bond is issued[4], recall the function H, one should not issue any CAT bond too close to the end, this allows to compensate it.

Given $\phi \in \Phi_\kappa^{z,\overline{m}}$, the expected gain is

$$J(z, p, m; \phi) := \mathbb{E}_{\overline{m}}[G^{z,p,m}(\phi)],$$

and

$$\mathrm{v}(z, p, m) := \sup_{\phi \in \Phi_\kappa^{z,\overline{m}}} J(z, p, m; \phi)$$

is the corresponding value function. Note that v is bounded.

3 Value function characterization

In order to introduce the PDE, we first need the definition of a new function. Recall the set $\mathcal{J}(c; u)$ defined in (2.5). Then,

$$\mathfrak{F}(z; u) := \sum_{j \in \mathcal{J}(c;u)} F(t, x, c^j, l^j; u), \quad z := (t, x, c, l),$$

represents the total *payoff* for the ends of the CAT bonds according to the jump u. Recall that $\Pi : \mathbf{C} \to \mathbf{K}$ gives the number of running policies where $\mathbf{K} := \{0, \ldots, \kappa\}$.

For ease of notation, we define $\mathbf{D} := [0, T] \times \mathbb{R}^d \times \mathbf{CL} \times \mathbf{P} \times \mathbf{M}$, and for $\mathbf{J} \in \mathcal{P}(\mathbf{K})$, $\mathbf{D_J} := [0, T] \times \mathbb{R}^d \times \mathbf{CL_J} \times \mathbf{P} \times \mathbf{M}$. To $\mathbf{J} \in \mathcal{P}(\mathbf{K})$, we denote by $\mathbf{1_J} = (\mathbf{1_J}(j))_{1 \le j \le \kappa}$ the vector in $\mathbb{R}^\kappa$ in which $\mathbf{1_J}(j) = 1$ if $j \in \mathbf{J}$, 0 else.

For $(z, p, m) \in \mathbf{D}$ and $u \in \mathbb{R}^{d*}$, we introduce the operator $\mathcal{I}$ defined, for all $(z, p, m) \in \mathbf{D}$, by:

$$\mathcal{I}[\varphi, u](z, p, m) := \varphi(t, x + \beta(t, x, u) + \mathfrak{F}(z; u), \mathfrak{C}_-(c, l; u), p + h_2(t, p), \mathfrak{M}_\gamma(m^\gamma; u), m^\upsilon).$$

Thus, the Dynkin operator associated with our problem with policies running in indexes $\mathbf{J}$ is:

[4] To issue a CAT bonds has a cost.

$$\mathcal{L}^{\mathbf{J}}\varphi := \partial_t\varphi + \langle \mu + \sum_{j=1}^{\kappa} \mathbf{1}_{\mathbf{J}}(j)\overline{C}(t,c^j), D\varphi\rangle + \langle \mathbf{1}_{\mathbf{J}}, D_l\varphi\rangle + \langle h_1, D_p\varphi\rangle +$$
$$\int_{\mathbb{R}^d} [\mathcal{I}[\varphi,u]-\varphi]\,\Lambda(t,\lambda_0)\Upsilon(\gamma_0,du),$$

in which recall that Υ denotes the size distribution of the jumps of the random Poisson measure N. Moreover, we introduce:

$$\mathcal{L}^{\mathbf{J}}_{\star}\varphi := \mathbb{E}_{\overline{m}}\left[\mathcal{L}^{\mathbf{J}}\varphi\right],$$

and

$$\mathbf{D}_{\circ} := [0,T) \times \mathbb{R}^d \times \mathbf{CL}_{\mathbf{J}} \times \mathbf{P} \times \mathbf{M},$$
$$\mathbf{D}_{T} := \{T\} \times \mathbb{R}^d \times \mathbf{CL} \times \mathbf{P} \times \mathbf{M}.$$

Then, we expect that v is a viscosity solution of, for each $\mathbf{J} \in \mathcal{P}(\mathbf{K})$ and non-empty $\mathbf{J}' \subset \mathbf{J}$,

$$\mathbf{1}_{\{\mathbf{J}=\mathbf{K}\}}\left[-\mathcal{L}^{\mathbf{K}}_{\star}\varphi\right] + \mathbf{1}_{\{\mathbf{J}\neq\mathbf{K}\}}\min\{-\mathcal{L}^{\mathbf{J}}_{\star}\varphi, \varphi - \mathcal{K}\varphi\} = 0 \quad \text{on } \mathbf{D}_{\circ} \tag{3.1}$$
$$\varphi = \mathbf{1}_{\{\mathbf{J}=\mathbf{K}\}}g + \mathbf{1}_{\{\mathbf{J}\neq\mathbf{K}\}}\max\{\mathcal{K}g, g\} \quad \text{on } \mathbf{D}_T \tag{3.2}$$
$$\lim_{l' \to \mathfrak{L}^{\mathbf{J}'}_{\mathbf{J}}(l)} \varphi(.,c,l',.) = \max\{\varphi(.,\mathfrak{C}^{\ell}_{-}[c,\mathfrak{L}^{\mathbf{J}'}_{\mathbf{J}}(l)],.), \mathcal{K}\varphi(.,\mathfrak{C}^{\ell}_{-}[c,\mathfrak{L}^{\mathbf{J}'}_{\mathbf{J}}(l)],.)\} \quad \text{on } \mathbf{D}\backslash\mathbf{D}_{\emptyset} \tag{3.3}$$

in which, for $(z,p,m) \in \mathbf{D}_{\mathbf{J}}$ and $\phi^a \in \Phi^{z,\overline{m}}$ a control such that $\{\tau_1^{\phi^a} = t, \alpha_1^{\phi^a} = a\}$ holds with probability one,

$$\mathcal{K}\varphi := \sup_{a\in\mathbf{A}} \mathcal{K}^a\varphi, \quad \mathcal{K}^a\varphi(z,p,m) := \mathbb{E}_{\overline{m}}[\varphi(Z^{z,\phi^a}_{t+}, p, M^{z,m,\phi^a}_{t+})];$$

and, for $\mathbf{J}' \subset \mathbf{J}$,

$$\mathfrak{L}^{\mathbf{J}'}_{\mathbf{J}} : [0,\ell]^{\mathbf{J}} \to [0,\ell]^{\mathbf{J}} \tag{3.4}$$
$$(l_j)_{1\le j\le\kappa} \mapsto (\ell\mathbf{1}_{\{j\in\mathbf{J}'\}} + l_j\mathbf{1}_{\{j\notin\mathbf{J}'\}})_{1\le j\le k}, \tag{3.5}$$

where $[0,\ell]^{\mathbf{J}} := \{l \in ([0,\ell]\cup\partial)^{\kappa} : l_j \neq \partial \Leftrightarrow j \in \mathbf{J}\}$.

Remark 3.1. *Note that the above corresponds to the definition of a system of PDEs linked by the common boundary conditions.*

We now define what is a viscosity solution of (3.1)-(3.2)-(3.3). For $\mathbf{J} \in \mathcal{P}(\mathbf{K})$, we define:

$$\mathscr{C}^1_{\mathbf{J}} := \left\{\varphi : \mathbf{D}_{\mathbf{J}} \mapsto \mathbb{R}, \ \ \varphi \in C^{1,1,(0,1),1,0}(\mathbf{D}_{\mathbf{J}})\right\}.$$

Definition 3.1. *We say that a upper-semicontinuous function u on $\mathbf{D}$ is a viscosity sub-solution of (3.1)-(3.2)-(3.3) if, for any $\mathbf{J} \in \mathcal{P}(\mathbf{K})$, $(z_\circ,p_\circ,m_\circ) \in \mathbf{D}_{\mathbf{J}}$, and $\varphi \in \mathscr{C}^1_{\mathbf{J}}$ such that $\max_{\mathbf{D}_{\mathbf{J}}}(u-\varphi) = (u-\varphi)(z_\circ,p_\circ,m_\circ) = 0$ we have, if $t_\circ < T$,*

$$\mathbf{1}_{\{\mathbf{J}=\mathbf{K}\}}\left[-\mathcal{L}^{\mathbf{K}}_{\star}\varphi\right] + \mathbf{1}_{\{\mathbf{J}\neq\mathbf{K}\}}\min\{-\mathcal{L}^{\mathbf{J}}_{\star}\varphi, \varphi - \mathcal{K}u)\}(z_\circ,p_\circ,m_\circ) \le 0,$$

if $\mathbf{J} \neq \emptyset$, for any non-empty $\mathbf{J}' \in \mathcal{P}(\mathbf{J})$, with $d_\circ = (t_\circ, x_\circ, c_\circ, \mathfrak{L}_{\mathbf{J}}^{\mathbf{J}'}(l_\circ), p_\circ, m_\circ)$ and $d'_\circ = (t_\circ, x_\circ, \mathfrak{C}_-^\ell[c_\circ, \mathfrak{L}_{\mathbf{J}}^{\mathbf{J}'}(l_\circ)], p_\circ, m_\circ)$,

$$\limsup_{(z,p,m)\to d_\circ} u(z,p,m) \leq \max\left\{u(d'_\circ), \mathcal{K}u(d'_\circ)\right\},$$

and, if $t_\circ = T$,

$$u(z_\circ, p_\circ, m_\circ) \leq \left\{\mathbf{1}_{\{\mathbf{J}=\mathbf{K}\}} g + \mathbf{1}_{\{\mathbf{J}\neq\mathbf{K}\}} \max(\mathcal{K}g, g)\right\}(x_\circ, c_\circ, l_\circ, p_\circ, m_\circ).$$

We say that a lower-semicontinuous function v on $\mathbf{D}$ is a viscosity super-solution of (3.1)-(3.2)-(3.3) if, for any $\mathbf{J} \in \mathcal{P}(\mathbf{K})$, $(z_\circ, p_\circ, m_\circ) \in \mathbf{D}_{\mathbf{J}}$, and $\varphi \in \mathscr{C}^1_{\mathbf{J}}$ such that $\min_{\mathbf{D}_{\mathbf{J}}}(v - \varphi) = (v-\varphi)(z_\circ, p_\circ, m_\circ) = 0$ we have, if $t_\circ < T$,

$$\mathbf{1}_{\{\mathbf{J}=\mathbf{K}\}}\left[-\mathcal{L}_\star^{\mathbf{K}}\varphi\right] + \mathbf{1}_{\{\mathbf{J}\neq\mathbf{K}\}} \min\{-\mathcal{L}_\star^{\mathbf{J}}\varphi, \varphi - \mathcal{K}v)\}(z_\circ, p_\circ, m_\circ) \geq 0,$$

if $\mathbf{J} \neq \emptyset$, for any non-empty $\mathbf{J}' \in \mathcal{P}(\mathbf{J})$, with $d_\circ = (t_\circ, x_\circ, c_\circ, \mathfrak{L}_{\mathbf{J}}^{\mathbf{J}'}(l_\circ), p_\circ, m_\circ)$ and $d'_\circ = (t_\circ, x_\circ, \mathfrak{C}_-^\ell[c_\circ, \mathfrak{L}_{\mathbf{J}}^{\mathbf{J}'}(l_\circ)], p_\circ, m_\circ)$,

$$\liminf_{(z,p,m)\to d_\circ} v(z,p,m) \geq \max\left\{v(d'_\circ), \mathcal{K}v(d'_\circ)\right\}$$

and, if $t_\circ = T$,

$$v(z_\circ, p_\circ, m_\circ) \geq \left\{\mathbf{1}_{\{\mathbf{J}=\mathbf{K}\}} g + \mathbf{1}_{\{\mathbf{J}\neq\mathbf{K}\}} \max(\mathcal{K}g, g)\right\}(x_\circ, c_\circ, l_\circ, p_\circ, m_\circ).$$

We say that a function u is a viscosity solution of (3.1)-(3.2)-(3.3) if its upper-semicontinuous envelope u^* is a viscosity sub-solution and its lower-semicontinuous envelope u_* is a viscosity super-solution of (3.1)-(3.2)-(3.3).

To ensure that the above operator is continuous, we first assume that:

Assumption 3.1. *$\mathcal{K}\varphi$ is upper- (resp. lower-) semicontinuous, for all upper- (resp. lower-) semicontinuous bounded function φ.*

A sufficient condition for Assumption 3.1 to hold is provided in [3], see the discussion after equation (3.6).

In order to ensure that $\mathcal{L}_*^{\mathbf{J}}$ is continuous for all $\mathbf{J} \in \mathcal{P}(\mathbf{K})$, we make the following assumption.

Assumption 3.2. *We assume that*

- *The functions F and $\mathfrak{M}_\gamma$ are continuous ;*
- *The stochastic kernel $\gamma \mapsto \Upsilon(\gamma, du)$ is continuous ;*
- *There map $(t,\lambda) \mapsto \Lambda(t,\lambda)$ is continuous.*

Lemma 3.1. *Assume that Assumption 3.2 holds. Then, for all $(c,m) \in \mathbf{C} \times \mathbf{M}$, with $\mathbf{J} := \{j \in \mathbf{K} : c_j \neq \partial\}$, and for all bounded function $\varphi \in \mathscr{C}^1_{\mathbf{J}}$, the operator $\mathcal{L}_\star^{\mathbf{J}}\varphi$ is continuous.*

Proof. Let $(c, m) \in \mathbf{C} \times \mathbf{M}$ and $\mathbf{J}$ defined as above. Recall that

$$\mathcal{L}_{\star}^{\mathbf{J}}\varphi = \partial_t \varphi + \langle \mu + \sum_{j=1}^{\kappa} \mathbf{1}_{\mathbf{J}}(j)\overline{C}(t, c^j), D\varphi \rangle + \langle \mathbf{1}_{\mathbf{J}}, D_l \varphi \rangle + \langle h_1, D_p \varphi \rangle$$
$$+ \mathbb{E}_{\overline{m}} \left[\int_{\mathbb{R}^d} [\mathcal{I}[\varphi, u] - \varphi] \, \Lambda(t, \lambda_0) \Upsilon(\gamma_0, du) \right].$$

For the first line above, since all involved functions are continuous, the operator is continuous. For the second line, since φ is bounded, one easily checks that the expected value with respect to (λ, γ) is well defined and one can apply Fubini's theorem. This is rewritten:

$$\overline{\Lambda}(t, p) \int_{U^\gamma} \left[\int_{\mathbb{R}^d} [\mathcal{I}[\varphi, u] - \varphi] \, \Upsilon(\gamma, du) \right] dm^\gamma(\gamma)$$

with $\overline{\Lambda}(t, p) := \int_{U^\lambda} \Lambda(t, \lambda) dm^\lambda(\lambda)$ which is continuous, see [13, Proposition 7.30 p145].

Now, remark that the function integrated through $\Upsilon(\gamma, du)$ with $\gamma \in U^\gamma$ fixed is continuous by definition. Since the stochastic kernel $\gamma \mapsto \Upsilon(\gamma, du)$ is assumed to be continuous, we get again from [Proposition 7.30 p145] that the function integrated through m^γ is continuous and bounded. And then, the operator is continuous. □

We now assume that we have a comparison principle. A sufficient condition is provided in Proposition 5.1 below.

Assumption 3.3. *Let U (resp. V) be a upper- (resp. lower-) semicontinuous bounded viscosity sub- (resp. super-) solution of (3.1)-(3.2)-(3.3). Assume further that $U \leq V$ on $\mathbf{D}_T$. Then, $U \leq V$ on $\mathbf{D}$.*

Theorem 3.1. *The function* v *is the unique viscosity solution of (3.1)-(3.2)-(3.3).*

4 Viscosity solution properties

This part is dedicated to the proof of the viscosity solution characterization of Theorem 3.1. We start with the sub-solution property and continue with the super-solution property. The main difficulty relies on the fact that the filtration depends on the initial data. The results can be obtained along the lines of [3].

4.1 Sub-solution property

Proposition 4.1. *The function* v *is a viscosity sub-solution of (3.1)-(3.2)-(3.3).*

The proof of this proposition, as usual, relies on a dynamic programming principle. For this part, the dependency of the filtration on the initial data in not problematic as it only requires a conditioning argument. We have the following result:

Proposition 4.2. *Fix* $\mathbf{J} \in \mathcal{P}(\mathbf{K})$ *and* $(z,p,m) \in \mathbf{D_J}$*, and let* θ *be the first exit time of* $(Z^{z,\phi^0}, P^{t,p})$ *from a Borel set* $B \subset \mathbf{D_J}$ *containing* (z,p,m) *where* $\phi^0 \in \Phi^{z,m}$ *is a control such that* $\tau_1^{\phi^0} > t$*. Then,*

$$\mathrm{v}(z,p,m) \leq \sup_{\phi \in \Phi^{z,m}_{\geq t}} \mathbb{E}_m \left[\mathrm{v}^*(Z^{z,\phi}_\theta, P^{t,p}_\theta, m)\mathbf{1}_{\{\theta < \tau_1^\phi\}} + \mathcal{K}^{\alpha_1^\phi} \mathrm{v}^*(Z^{z,\phi}_{\tau_1^\phi -}, P^{t,p}_{\tau_1^\phi -}, m)\mathbf{1}_{\{\theta \geq \tau_1^\phi\}} \right] \tag{4.1}$$

in which $z := (t,x,c,l)$*,* $\Phi^{z,\overline{m}}_{\geq t} := \{\phi \in \Phi^{z,\overline{m}}_\kappa : \tau_1^\phi \geq t\}$*.*

Proof. It suffices to follow the arguments of Proposition 4.2 in [3]. □

We now prove Proposition 4.1.

Proof. Since, for each $\mathbf{J} \in \mathcal{P}(\mathbf{K})$, the operator $\mathcal{L}^{\mathbf{J}}_\star$ is continuous, the proof of (3.1) and (3.2) can be obtained by using the same arguments as in Proposition 4.1 in [3].

To prove (3.3), one can use the same arguments used in order to prove (3.2). □

4.2 Super-solution property

Because of the non-trivial dependence of the filtration $\mathbb{F}^{z,m,\phi}$ with respect to the initial data, in order to prove the super-solution property associated with Theorem 3.1, we shall use a discrete version of our impulse control problem, as in [3]. We shall show that the limit problem is a super-solution of (3.1)-(3.2)-(3.3). Proposition 4.1 and the comparison assumption will show that the limit problem is v.

We shall use a dynamic programing principle in some discrete form defined below.

Proposition 4.3. *Fix* $\mathbf{J} \in \mathcal{P}(\mathbf{K})$ *and* $(z,p,m) \in \mathbf{D_J}$*. Let* $\Phi^{z,\overline{m}}_n$ *be the subset of elements of* $\Phi^{z,\overline{m}}_\kappa$ *such that the stopping times* $\tau_i^\phi, i \geq 1$ *are valued in* $\{t\} \cup \pi_n \cap [t,T]$ *with* $\pi_n := \{kT/2^n; 0 \leq k \leq 2^n\}$*. The corresponding value function is:*

$$\mathrm{v}_n(z,p,m) := \sup_{\phi \in \Phi^{z,\overline{m}}_n} J(z,p,m;\phi), \quad (z,p,m) \in \mathbf{D}.$$

Let $(\theta^\phi, \phi \in \Phi^{z,m}_n)$ *be such that* θ^ϕ *is a* $\mathbb{F}^{z,\overline{m},\phi}$*-stopping time valued in* $\{t\} \cup \pi_n \cap [t,T]$*. Then,*

$$\mathrm{v}_n(z,p,m) = \sup_{\phi \in \Phi^{z,\overline{m}}_n} \mathbb{E}_{\overline{m}} \left[\mathrm{v}_n(Z^{z,\phi}_{\theta^\phi}, P^{t,p}_{\theta^\phi}, M^{z,m,\phi}_{\theta^\phi}) \right].$$

Proof. It suffices to follow the arguments of Lemma 4.1, Proposition 4.3 and Corollary 4.1 in [3]. □

We now consider the limit $n \to +\infty$. Let us set, for $(z,p,m) \in \mathbf{D}$,

$$\mathrm{v}_\circ(z,p,m) := \liminf_{(z',p',m',n) \to (z,p,m,+\infty)} \mathrm{v}_n(z',p',m').$$

Proposition 4.4. *The function* $\mathrm{v}_\circ$ *is a viscosity super-solution of (3.1)-(3.2)-(3.3).*

Proof. The equations (3.1) and (3.2) can be obtained by using Proposition 4.3 and following the arguments in the proof of Proposition 4.4 in [3].

We now prove the boundary condition (3.3).

Step 1. Fix $\mathbf{J} \subset \mathcal{P}(\mathbf{K})$ and $(z,p,m) \in \mathbf{D_J}$.

Let $n_k \to +\infty$ and $(z_k,p_k,m_k) \to (z,p,m)$ such that $\mathrm{v}_{n_k}(z_k,p_k,m_k) \to \mathrm{v}_\circ(z,p,m)$. Let $k_\circ \geq 1$ and define the lower semi-continuous function $\varphi_{k_\circ}$ as in the proof of Proposition 4.4 in [3]. Then, from Proposition 4.3, with $\phi^0 \in \Phi^{t,x,m}$ a control such that $\tau_1^{\phi^0} > T$, we get for $k \geq k_\circ$

$$\mathrm{v}_{n_k}(z_k,p_k,m_k) \geq \mathbb{E}_{\overline{m}}\left[\varphi_{k_\circ}(Z^{z_k,\phi^0}_{\theta^{\phi^0}}, P^{t_k,p_k}_{\theta^{\phi^0}}, M^{z_k,m_k,\phi^0}_{\theta^{\phi^0}})\right].$$

Then, $k \to +\infty$ leads to

$$\mathrm{v}_\circ(z,p,m) \geq \mathbb{E}_{\overline{m}}\left[\varphi_{k_\circ}(Z^{z,\phi_0}_{\theta^{\phi^0}}, P^{t,p}_{\theta^{\phi^0}}, M^{z,m,\phi^0}_{\theta^{\phi^0}})\right]$$

and, again from the proof of Proposition 4.4 in [3], we get that $\lim_{k_\circ \to +\infty} \varphi_{k_\circ} \geq \mathrm{v}_\circ$. By Fatou's lemma we have

$$\mathrm{v}_\circ(z,p,m) \geq \mathbb{E}_{\overline{m}}\left[\mathrm{v}_\circ(Z^{z,\phi_0}_{\theta^{\phi^0}}, P^{t,p}_{\theta^{\phi^0}}, M^{z,m,\phi^0}_{\theta^{\phi^0}})\right].$$

Step 2. Now fix $\mathbf{J'} \subset \mathbf{J}$ and $(z_\circ,p_\circ,m_\circ) \in \mathbf{D_J}$. Let $k \to +\infty$ and $(z_k,p_k,m_k) \to (t_\circ,x_\circ,c_\circ,\mathfrak{L}^{\mathbf{J'}}_{\mathbf{J}}(l_\circ),p_\circ,m_\circ)$ such that

$$\mathrm{v}_\circ(z_k,p_k,m_k) \to \liminf_{(z,p,m)\to(t_\circ,x_\circ,c_\circ,\mathfrak{L}^{\mathbf{J'}}_{\mathbf{J}}(l_\circ),p_\circ,m_\circ)} \mathrm{v}_\circ(z,p,m).$$

We introduce $h_k := \mathfrak{L}^{\mathbf{J'}}_{\mathbf{J}}(l_\circ) - l_k$. Then, for $k_\circ$ large enough, we can find $\varepsilon > 0$ such that $\sup_{k \geq k_\circ} \max_{j \in \mathbf{J'}} h^j_k < \varepsilon < \inf_{k \geq k_\circ} \max_{j \in \mathbf{J}\setminus\mathbf{J'}} (\ell - l^j_k)$. Then, for $k \geq k_\circ$,

$$\mathrm{v}_\circ(t_k,x_k,c_k,\mathfrak{L}^{\mathbf{J'}}_{\mathbf{J}}(l_\circ) - h_k,p_k,m_k) \geq \mathbb{E}\left[\mathrm{v}_\circ(Z^{z_k,\phi_0}_{t+\varepsilon}, P^{t_k,p_k}_{t+\varepsilon}, M^{z_k,m_k,\phi_0}_{t+\varepsilon})\right].$$

Now, we send $k \to +\infty$, since the functions in the diffusion are Lipschitz, using Fatou's lemma leads to

$$\lim_{k\to+\infty} \mathrm{v}_\circ(t_k,x_k,c_k,\mathfrak{L}^{\mathbf{J'}}_{\mathbf{J}}(l_\circ) - h_k,p_k,m_k) \geq \mathbb{E}\left[\mathrm{v}_\circ(Z^{z,\phi_0}_{t+\varepsilon}, P^{t,p}_{t+\varepsilon}, M^{z,m,\phi_0}_{t+\varepsilon})\right].$$

Since, under the control ϕ^0, the processes X, P and M are driven here by the random Poisson measure with finite activity, they satisfy the stochastic continuity property. Moreover, since the probability of observing a jump decreases to 0 when $\varepsilon \to 0$, one easily shows that,

$$\lim_{k\to+\infty} \mathrm{v}_\circ(t_k,x_k,c_k,\mathfrak{L}^{\mathbf{J'}}_{\mathbf{J}}(l_\circ) - h_k,p_k,m_k) \geq \mathrm{v}_\circ(t_\circ,x_\circ,\mathfrak{C}^{\ell}_{-}[c_\circ,\mathfrak{L}^{\mathbf{J'}}_{\mathbf{J}}(l_\circ)],p_\circ,m_\circ),$$

by using the fact that $\mathrm{v}_\circ$ is bounded and the definition of the process C and L after the end of one or several CAT bonds.

Step 3. In order to show the second inequality, repeat Step 1. and Step 2. using, instead of ϕ^0, a control $\phi^a \in \Phi_\kappa^{z,\overline{m}}$ such that $\{\tau_1^{\phi^a} = t, \alpha_1^{\phi^a} = a, \tau_2^{\phi^a} > T\}$ holds with probability one. $\square$

We now prove Theorem 3.1.

Proof of Theorem 3.1. We already know that v^* and $\mathrm{v}_\circ$ are respectively a bounded sub- and super-solution of (3.1)-(3.2)-(3.3). Then, under Assumption 3.3, $\mathrm{v}^* \leq \mathrm{v}_\circ$. Moreover, by construction, $\mathrm{v}_\circ \leq \mathrm{v} \leq \mathrm{v}^*$. Then, v is continuous and the unique solution of (3.1)-(3.2)-(3.3). $\square$

Remark 4.1. *If we denote by $\mathcal{S}_{\mathbf{K}}$ the set of permutation of $\{1 \leq k \leq \kappa\}$, then, by symmetry,*

$$\mathrm{v}(z, p, m) = \mathrm{v}(t, x, (c, l) \circ \Sigma, p, m)$$

for each $\Sigma \in \mathcal{S}_{\mathbf{K}}$, $(z, p, m) \in \mathbf{D}$. From a numerical point of view, this allows to only compute the value function on $\kappa + 1$ different dimensions for the CAT bonds space $\mathbf{CL}$ on which we can order them, instead of 2^κ different dimensions with no order.

5 A sufficient condition for the comparison

In this section, we provide a sufficient condition for Assumption 3.3 to hold.

Proposition 5.1. *Assumption 3.3 holds whenever there exists a function Ψ on $[0, T) \times \mathbb{R}^d \times \mathbf{CL} \times \mathbf{P} \times \mathbf{M}$ such that, for each $\mathbf{J} \in \mathcal{P}(\mathbf{K})$,*

(i) $(t, x, l, p) \mapsto \Psi(t, x, c, l, p, m) \in C^{1,1,1,1}([0, T) \times \mathbb{R}^d \times [0, \ell) \times \mathbf{P})$ *for all* $(c, m) \in \mathbf{C} \times \mathbf{M}$,

(ii) $\varrho\Psi \geq \mathcal{L}_*^{\mathbf{J}}\Psi$ *on* $\mathbf{D}_{\mathbf{J}}$ *for some* $\varrho > 0$,

(iii) $\Psi - \mathcal{K}\Psi \geq \delta$ *on* $\mathbf{D}_{\mathbf{J}}$ *for some* $\delta > 0$,

(iv) $\Psi \geq max(\mathcal{K}\tilde{g}, \tilde{g})$ *on* $\mathbb{R}^d \times \mathbf{CL}_{\mathbf{J}} \times \mathbf{P} \times \mathbf{M}$ *with* $\tilde{g}(t, \cdot) := e^{\varrho t} g(t, \cdot)$ *and* ϱ *is defined in (ii),*

(v) $\liminf_{l' \to \mathfrak{L}_{\mathbf{J}}^{\mathbf{J}'}(l)} \Psi(\cdot, c, l', \cdot) - \Psi(\cdot, \mathfrak{C}_-^\ell(c, \mathfrak{L}_{\mathbf{J}}^{\mathbf{J}'}(l)), .) \geq 0$ *for all* $\mathbf{J}' \subset \mathbf{J}$,

(vi) $\Psi^- \leq \overline{\Psi}(x) = o(\|x\|^2)$ *as* $\|x\|^2 \to +\infty$ *for some* $\overline{\Psi} : \mathbb{R}^d \to \mathbb{R}$.

Proof. Step 1. As usual, we shall argue by contradiction. We assume that there exists some $\mathbf{J}_0 \in \mathcal{P}(\mathbf{K})$ and some $(z_0, p_0, m_0) \in \mathbf{D}_{\mathbf{J}}$ such that $(U - V)(z_0, p_0, m_0) > 0$, in which U is a sub-solution of (3.1)-(3.2)-(3.3) and V is a super-solution of (3.1)-(3.2)-(3.3). Recall the definition of Ψ, ϱ and $\tilde{g}$ in Proposition 5.1. We set $\tilde{u}(t, .) = e^{\varrho t} U(t, .)$ and $\tilde{v}(t, .) = e^{\varrho t} V(t, .)$ for all $(t, .) \in \mathbf{D}_{\mathbf{J}}$ for all $\mathbf{J} \in \mathcal{P}(\mathbf{K})$. Then, there exists $\lambda \in (0, 1)$ such that

$$(\tilde{u} - \tilde{v}^\lambda)(z_0, p_0, m_0) > 0, \tag{5.1}$$

in which $\tilde v^\lambda := (1-\lambda)\tilde v + \lambda\Psi$. Note that $\tilde u$ and $\tilde v$ are sub and super-solution on $\mathbf{D_J}$ of

$$\min\left\{\varrho\varphi - \mathcal{L}^{\mathbf{J}}_{*}\varphi, \varphi - \mathcal{K}\varphi\right\} = 0$$

for each $\mathbf{J} \in \mathcal{P}(\mathbf{K})$, with the boundary conditions

$$\mathbf{1}_{\{\mathbf{J}=\mathbf{K}\}}(\varphi(T,\cdot) - \tilde g) + \mathbf{1}_{\{\mathbf{J}\neq\mathbf{K}\}}\min\left\{\varphi(T,\cdot) - \tilde g, \varphi(T,\cdot) - \mathcal{K}\tilde g\right\} = 0, \tag{5.2}$$

and

$$\lim_{l'\to\mathfrak{L}^{\mathbf{J}'}_{\mathbf{J}}(l)} \varphi(.,c,l',.) = \varphi(.,\mathfrak{C}^{\ell}_{-}[c,\mathfrak{L}^{\mathbf{J}'}_{\mathbf{J}}(l)],.) \quad \forall \mathbf{J}' \subset \mathbf{J},\ (c,l) \in \mathbf{CL_J} \tag{5.3}$$

Step 2. Let $d_{\mathbf{M}}$ be a metric on $\mathbf{M}$ compatible with the weak topology. For $(t,x,y,c,l,p,q,m) \in \mathbf{D}' := [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbf{CL} \times \mathbf{P}^2 \times \mathbf{M}$, we set :

$$\begin{aligned}\Gamma_\varepsilon(t,x,y,c,l,p,q,m) :=& \tilde u(t,x,c,l,p,m) - \tilde v^\lambda(t,y,c,l,q,m)\\ &- \varepsilon\left(\|x\|^2 + \|y\|^2 + \mathfrak{d}(c,l) + \|p\|^2 + \|q\|^2 + d_{\mathbf{M}}(m)\right)\end{aligned} \tag{5.4}$$

with $\varepsilon > 0$ small enough such that $\Gamma_\varepsilon(t_0,x_0,x_0,c_0,l_0,p_0,p_0,m_0) > 0$. Although $[0,\ell)$ is not closed, note that the supremum is achieved for some $\mathbf{J}_\varepsilon \in \mathcal{P}(\mathbf{K})$ by some $(t_\varepsilon,x_\varepsilon,y_\varepsilon, c_\varepsilon,l_\varepsilon,p_\varepsilon,q_\varepsilon,m_\varepsilon) \in \mathbf{D}_{\mathbf{J}_\varepsilon}$. This follows from the upper-semicontinuity of Γ_ε, the fact that $\tilde u, -\tilde v$ and $-\Psi$ are bounded from above, and by the fact that

$$\limsup_{l'\to\mathfrak{L}^{k}_{\mathbf{J}}(l)} (\tilde u - \tilde v^\lambda)(.,c,l',.) \le (\tilde u - \tilde v^\lambda)(.,\mathfrak{C}^{\ell}_{-}(c,\mathfrak{L}^{k}_{\mathbf{J}}(l)),.).$$

For $(t,x,y,c,l,p,q,m) \in \mathbf{D}'$, we set

$$\Theta^n_\varepsilon(t,x,y,c,l,p,q,m) = \Gamma_\varepsilon(t,x,y,c,l,p,q,m) - n\left(\|x-y\|^2 + \|p-q\|^2\right).$$

Again, there is $(t^\varepsilon_n,x^\varepsilon_n,y^\varepsilon_n,c^\varepsilon_n,l^\varepsilon_n,p^\varepsilon_n,q^\varepsilon_n,m^\varepsilon_n) \in \mathbf{D}'$ such that

$$\sup_{\mathbf{D}'}\Theta^n_\varepsilon = \Theta^n_\varepsilon(t^\varepsilon_n,x^\varepsilon_n,y^\varepsilon_n,c^\varepsilon_n,l^\varepsilon_n,p^\varepsilon_n,q^\varepsilon_n,m^\varepsilon_n).$$

It is standard to show that, after possibly considering a subsequence,

$$\begin{aligned}&(t^\varepsilon_n,x^\varepsilon_n,y^\varepsilon_n,c^\varepsilon_n,l^\varepsilon_n,p^\varepsilon_n,q^\varepsilon_n,m^\varepsilon_n) \to (\hat t_\varepsilon,\hat x_\varepsilon,\hat y_\varepsilon,\hat c_\varepsilon,\hat l_\varepsilon,\hat p_\varepsilon,\hat q_\varepsilon,\hat m_\varepsilon) \in \mathbf{D}',\\ &n\left(\|x^\varepsilon_n - y^\varepsilon_n\|^2 + \|p^\varepsilon_n - q^\varepsilon_n\|^2\right) \to 0, \text{ and}\\ &\Theta^n_\varepsilon(t^\varepsilon_n,x^\varepsilon_n,y^\varepsilon_n,c^\varepsilon_n,l^\varepsilon_n,p^\varepsilon_n,q^\varepsilon_n,m^\varepsilon_n) \to \Gamma_\varepsilon(\hat t_\varepsilon,\hat x_\varepsilon,\hat y_\varepsilon,\hat c_\varepsilon,\hat l_\varepsilon,\hat p_\varepsilon,\hat q_\varepsilon,\hat m_\varepsilon) = \Gamma_\varepsilon(t_\varepsilon,x_\varepsilon,y_\varepsilon,c_\varepsilon,l_\varepsilon,p_\varepsilon,q_\varepsilon,m_\varepsilon),\end{aligned} \tag{5.5}$$

see e.g. [6, Lemma 3.1]. Moreover, up to a subsequence, there exists $n_0 \in \mathbb{N}$, such that, for all $n \ge n_0$, $(t^\varepsilon_n,x^\varepsilon_n,c^\varepsilon_n,l^\varepsilon_n,p^\varepsilon_n,m^\varepsilon_n) \in D_{\mathbf{J}_\varepsilon}$ and $(t^\varepsilon_n,y^\varepsilon_n,c^\varepsilon_n,l^\varepsilon_n,q^\varepsilon_n,m^\varepsilon_n) \in D_{\mathbf{J}_\varepsilon}$.

Step 3. We first assume that, up to a subsequence, $(\tilde u - \mathcal{K}\tilde u)(t^\varepsilon_n,x^\varepsilon_n,c^\varepsilon_n,l^\varepsilon_n,p^\varepsilon_n,m^\varepsilon_n) \le 0$, for $n \ge 1$. Then, it follows from the supersolution property of $\tilde v$ and Condition (iii) that

$$\begin{aligned}&\tilde u(t^\varepsilon_n,x^\varepsilon_n,c^\varepsilon_n,l^\varepsilon_n,p^\varepsilon_n,m^\varepsilon_n) - \tilde v^\lambda(t^\varepsilon_n,y^\varepsilon_n,c^\varepsilon_n,l^\varepsilon_n,q^\varepsilon_n,m^\varepsilon_n) \le\\ &\mathcal{K}\tilde u(t^\varepsilon_n,x^\varepsilon_n,c^\varepsilon_n,l^\varepsilon_n,p^\varepsilon_n,m^\varepsilon_n) - \mathcal{K}\tilde v^\lambda(t^\varepsilon_n,y^\varepsilon_n,c^\varepsilon_n,l^\varepsilon_n,q^\varepsilon_n,m^\varepsilon_n) - \lambda\delta.\end{aligned}$$

Passing to the lim sup and using (5.5) and (3.1), we obtain

$$(\tilde{u} - \tilde{v}^\lambda)(\hat{t}_\varepsilon, \hat{x}_\varepsilon, \hat{c}_\varepsilon, \hat{l}_\varepsilon, \hat{p}_\varepsilon, \hat{m}_\varepsilon) + \lambda\delta \le \mathcal{K}(\tilde{u} - \tilde{v}^\lambda)(\hat{t}_\varepsilon, \hat{x}_\varepsilon, \hat{c}_\varepsilon, \hat{l}_\varepsilon, \hat{p}_\varepsilon, \hat{m}_\varepsilon)$$

Now let us observe that

$$\begin{aligned}\sup_{\mathbf{D}}(\tilde{u} - \tilde{v}^\lambda) &= \lim_{\varepsilon\to 0} \sup_{(t,x,c,l,p,m)\in\mathbf{D}} \Gamma_\varepsilon(t, x, x, c, l, p, p, m) \\ &= \lim_{\varepsilon\to 0} \Gamma_\varepsilon(t_\varepsilon, x_\varepsilon, x_\varepsilon, c_\varepsilon, l_\varepsilon, p_\varepsilon, p_\varepsilon, m_\varepsilon) \\ &= \lim_{\varepsilon\to 0} \Gamma_\varepsilon(\hat{t}_\varepsilon, \hat{x}_\varepsilon, \hat{x}_\varepsilon, \hat{c}_\varepsilon, \hat{l}_\varepsilon, \hat{p}_\varepsilon, \hat{p}_\varepsilon, \hat{m}_\varepsilon),\end{aligned} \tag{5.6}$$

in which the last identity follows from (5.5). Combined with the above inequality, this shows that $\sup_{\mathbf{D}}(\tilde{u} - \tilde{v}^\lambda) + \lambda\delta \le \limsup_{\varepsilon\to 0} \mathcal{K}(\tilde{u} - \tilde{v}^\lambda)(\hat{t}_\varepsilon, \hat{x}_\varepsilon, \hat{c}_\varepsilon, \hat{l}_\varepsilon, \hat{p}_\varepsilon, \hat{m}_\varepsilon)$, which leads to a contradiction for ε small enough.

Step 4. We now show that there is a subsequence such that $t_n^\varepsilon < T$ for all $n \ge 1$. If not, one can assume that $t_n^\varepsilon = T$. If, up to a subsequence, one can have $\tilde{u}(T, x_n^\varepsilon, c_n^\varepsilon, l_n^\varepsilon, p_n^\varepsilon, m_n^\varepsilon)) \le \tilde{g}(T, x_n^\varepsilon, c_n^\varepsilon, l_n^\varepsilon, p_n^\varepsilon, m_n^\varepsilon)$, then it follows from (5.2) and Condition (iv) that,

$$\tilde{u}(T, x_n^\varepsilon, c_n^\varepsilon, l_n^\varepsilon, p_n^\varepsilon, m_n^\varepsilon)) - \tilde{v}^\lambda(T, y_n^\varepsilon, c_n^\varepsilon, l_n^\varepsilon, q_n^\varepsilon, m_n^\varepsilon) \le \tilde{g}(T, x_n^\varepsilon, c_n^\varepsilon, l_n^\varepsilon, p_n^\varepsilon, m_n^\varepsilon) - \tilde{g}(T, y_n^\varepsilon, c_n^\varepsilon, l_n^\varepsilon, q_n^\varepsilon, m_n^\varepsilon).$$

Hence,

$$\Gamma_\varepsilon(t_\varepsilon, x_\varepsilon, x_\varepsilon, c_\varepsilon, l_\varepsilon, p_\varepsilon, p_\varepsilon, m_\varepsilon) \le \tilde{g}(T, x_n^\varepsilon, c_n^\varepsilon, l_n^\varepsilon, p_n^\varepsilon, m_n^\varepsilon) - \tilde{g}(T, y_n^\varepsilon, c_n^\varepsilon, l_n^\varepsilon, q_n^\varepsilon, m_n^\varepsilon),$$

and (5.5) with (5.6) leads to $\sup_{\mathbf{D}}(\tilde{u} - \tilde{v}^\lambda) \le 0$, a contradiction. If, up to a subsequence, $\tilde{u}(T, x_n^\varepsilon, c_n^\varepsilon, l_n^\varepsilon, p_n^\varepsilon, m_n^\varepsilon) \le \mathcal{K}\tilde{g}(T, x_n^\varepsilon, c_n^\varepsilon, l_n^\varepsilon, p_n^\varepsilon, m_n^\varepsilon)$, by (5.2) and Condition (iv),

$$\tilde{u}(T, x_n^\varepsilon, c_n^\varepsilon, l_n^\varepsilon, p_n^\varepsilon, m_n^\varepsilon) - \tilde{v}^\lambda(T, y_n^\varepsilon, c_n^\varepsilon, l_n^\varepsilon, q_n^\varepsilon, m_n^\varepsilon) \le \mathcal{K}\tilde{g}(T, x_n^\varepsilon, c_n^\varepsilon, l_n^\varepsilon, p_n^\varepsilon, m_n^\varepsilon) - \mathcal{K}\tilde{g}(T, y_n^\varepsilon, c_n^\varepsilon, l_n^\varepsilon, q_n^\varepsilon, m_n^\varepsilon).$$

Hence,

$$\Gamma_\varepsilon(t_\varepsilon, x_\varepsilon, x_\varepsilon, c_\varepsilon, l_\varepsilon, p_\varepsilon, p_\varepsilon, m_\varepsilon) \le \mathcal{K}\tilde{g}(T, x_n^\varepsilon, c_n^\varepsilon, l_n^\varepsilon, p_n^\varepsilon, m_n^\varepsilon) - \mathcal{K}\tilde{g}(T, y_n^\varepsilon, c_n^\varepsilon, l_n^\varepsilon, q_n^\varepsilon, m_n^\varepsilon),$$

and combining Assumption 3.1 with (5.5) and (5.6) leads to $\sup_{\mathbf{D}}(\tilde{u} - \tilde{v}^\lambda) \le 0$, the same contradiction.

Step 5. In view of step 2, 3, 4, one can assume that $t_n^\varepsilon < T$, $(\tilde{u} - \mathcal{K}\tilde{u})(t_n^\varepsilon, x_n^\varepsilon, c_n^\varepsilon, l_n^\varepsilon, p_n^\varepsilon, m_n^\varepsilon) > 0$ and $(c_n^\varepsilon, l_n^\varepsilon) \in \mathbf{CL}_{\mathbf{J}^\varepsilon}$ for all $n \ge 1$. Using Ishii's Lemma and following standard arguments, see Theorem 8.3 and the discussion after Theorem 3.2 in [6], we deduce from the sub- and supersolution viscosity solutions property of $\tilde{u}$ and $\tilde{v}^\lambda$, and the Lipschitz continuity assumptions on μ, σ and β, that

$$\begin{aligned}&\varrho\left(\tilde{u}(t_n^\varepsilon, x_n^\varepsilon, c_n^\varepsilon, l_n^\varepsilon, p_n^\varepsilon, m_n^\varepsilon) - \tilde{v}^\lambda(t_n^\varepsilon, y_n^\varepsilon, c_n^\varepsilon, l_n^\varepsilon, q_n^\varepsilon, m_n^\varepsilon)\right) \le \\ &C\left(n(\|x_n^\varepsilon - y_n^\varepsilon\|^2 + \|p_n^\varepsilon - q_n^\varepsilon\|^2) + \varepsilon(1 + \|x_n^\varepsilon\|^2 + \|y_n^\varepsilon\|^2 + \|p_n^\varepsilon\|^2 + \|q_n^\varepsilon\|^2)\right),\end{aligned}$$

for some $C > 0$, independent of n and ε. In view of (5.4) and (5.5), we get

$$\varrho\Gamma_\varepsilon(\hat{t}_\varepsilon, \hat{x}_\varepsilon, \hat{x}_\varepsilon, \hat{c}_\varepsilon, \hat{l}_\varepsilon, \hat{p}_\varepsilon, \hat{p}_\varepsilon, \hat{m}_\varepsilon) \le 2C\varepsilon\left(1 + \|\hat{x}_\varepsilon\|^2 + \|\hat{p}_\varepsilon\|^2\right). \tag{5.7}$$

We shall prove in next step that the right-hand side of (5.7) goes to 0 as $\varepsilon \to 0$, up to a subsequence. Combined with (5.6), this leads to a contradiction of (5.1).

Step 6. We conclude the proof by proving the claim used above. First note that we can always construct a sequence $(\tilde{t}_\varepsilon, \tilde{x}_\varepsilon, \tilde{c}_\varepsilon, \tilde{l}_\varepsilon, \tilde{p}_\varepsilon, \tilde{m}_\varepsilon)_\varepsilon)_{\varepsilon>0}$ such that

$$\Gamma_\varepsilon(\tilde{t}_\varepsilon, \tilde{x}_\varepsilon, \tilde{x}_\varepsilon, \tilde{c}_\varepsilon, \tilde{l}_\varepsilon, \tilde{p}_\varepsilon, \tilde{p}_\varepsilon, \tilde{m}_\varepsilon) \to \sup_{\mathbf{D}}(\tilde{u} - \tilde{v}^\lambda) \text{ and}$$
$$\varepsilon \left(\|\tilde{x}_\varepsilon\|^2 + \mathfrak{d}(\tilde{c}_\varepsilon, \tilde{l}_\varepsilon) + \|\tilde{p}_\varepsilon\|^2 + d_{\mathbf{M}}(\tilde{m}_\varepsilon) \right) \to 0 \quad \text{as } \varepsilon \to 0.$$

By (5.5), $\Gamma_\varepsilon(\tilde{t}_\varepsilon, \tilde{x}_\varepsilon, \tilde{x}_\varepsilon, \tilde{c}_\varepsilon, \tilde{l}_\varepsilon, \tilde{p}_\varepsilon, \tilde{p}_\varepsilon, \tilde{m}_\varepsilon) \le \Gamma_\varepsilon(\hat{t}_\varepsilon, \hat{x}_\varepsilon, \hat{x}_\varepsilon, \hat{c}_\varepsilon, \hat{l}_\varepsilon, \hat{p}_\varepsilon, \hat{p}_\varepsilon, \hat{m}_\varepsilon)$. Hence, $\sup_{\mathbf{D}}(\tilde{u} - \tilde{v}^\lambda) \le \sup_{\mathbf{D}}(\tilde{u} - \tilde{v}^\lambda) - 2 \liminf_{\varepsilon \to 0} \varepsilon \left(\|\hat{x}_\varepsilon\|^2 + \|\hat{p}_\varepsilon\|^2 \right)$.

□

6 Numerical Scheme

We let $h_\circ$ be a time-discretization step such that both $T/h_\circ$ and $\ell/h_\circ$ are an integer. In order to ensure the existence of such a $h_\circ$, we shall assume that $(T/\ell) \in \mathbb{Q}_+^*$ which does not appear as a restriction from a practical point of view. We set $\mathbf{T}^{h_\circ} := \{t_i^{h_\circ} := ih_\circ, i \le T/h_\circ\}$ and, for $\mathbf{J} \in \mathcal{P}(\mathbf{K})$, we set $\mathbf{L}_{\mathbf{J}}^{h_\circ} = \prod_{j=1}^{\kappa}(\partial \mathbf{1}_{\mathbf{J}^c}(j) + \mathbf{L}^{h_\circ}\mathbf{1}_{\mathbf{J}}(j))$ in which $\mathbf{L}^{h_\circ} := \{l_i^{h_\circ} := ih_\circ, i < \ell/h_\circ\}$.

The space $\mathbb{R}^d$ is discretized with a space step $h_\star$ on a rectangle $[-c, c]^d$ containing $N_{h_\star}^x$ points on each direction. The corresponding set is denoted by $\mathbf{X}_c^{h_\star}$. Recall that $\mathbf{P}$ is a subset of $\mathbb{R}^d$. We again discretise $\mathbb{R}^d$ with the same step space $h_\star$ on a rectangle $[-c, c]^d$ containing $N_{h_\star}^p$ points. The corresponding set is denoted by $\mathbf{P}_c^{\circ, h_\star}$, thus, the discretization of $\mathbf{P}$ is $\mathbf{P}_c^{h_\star} := \mathbf{P}_c^{\circ, h_\star} \cap \mathbf{P}$.

We set $h = (h_\circ, h_\star)$. The first order derivatives $(\partial_t \varphi)$, $(\partial_{x_i}\varphi)_{i \le d}$, $(\partial_{l_i}\varphi)_{i \le \kappa}$ and $(\partial_{p_i}\varphi)_{i \le d}$ are approximated by using the standard up-wind approximations:

$$\begin{aligned}
\Delta_i^{h_\circ, t}\varphi(z, p, m) &:= h_\circ^{-1}\left(\varphi(t + h_\circ, \cdot) - \varphi\right) \\
\Delta_i^{h_\star, x}\varphi(z, p, m) &:= \begin{cases} h_\star^{-1}\left(\varphi(\cdot, x + e_i h_\star, \cdot) - \varphi\right) & \text{if } \mu_i + \sum_{j=1}^{\kappa} \overline{C} \ge 0 \\ h_\star^{-1}\left(\varphi - \varphi(\cdot, x - e_i h_\star, \cdot)\right) & \text{else} \end{cases} \\
\Delta_i^{h_\star, \ell}\varphi(z, p, m) &:= \begin{cases} h_\star^{-1}\left(\varphi(\cdot, l + e_i h_\star, \cdot) - \varphi\right) & \text{if } i \in \mathbf{J} \\ 0 & \text{else} \end{cases} \\
\Delta_i^{h_\star, p}\varphi(z, p, m) &:= \begin{cases} h_\star^{-1}\left(\varphi(\cdot, p + e_i h_\star, \cdot) - \varphi\right) & \text{if } h_1 \ge 0 \\ h_\star^{-1}\left(\varphi - \varphi(\cdot, p - e_i h_\star, \cdot)\right) & \text{else} \end{cases}
\end{aligned}$$

in which e_i is $i - th$ unit vector of $\mathbb{R}^d$.

We shall assume that $\mathbf{A}$ is finite. We introduce:

$$\mathbf{C}_{\mathbf{J}}^{h_\star} := \prod_{j=1}^{\kappa}(\partial \mathbf{1}_{\mathbf{J}^c}(j) + (\mathbf{X}_c^{h_\star} \times \mathbf{R}_c^{h_\star} \times \mathbf{A})\mathbf{1}_{\mathbf{J}}(j)),$$

in which $\mathbf{R}_c^{h_\star} := \{ih_\star : -c/h_\star \le i \le c/h_\star\}$.

Then, the discrete counter-part of the set of policies running in indexes $\mathbf{J}$ is defined by

$$\mathbf{CL}^h_{\mathbf{J}} := \mathbf{C}^{h_\star}_{\mathbf{J}} \times \mathbf{L}^{h_\circ}_{\mathbf{J}}.$$

We introduce:

$$\overline{\Lambda}[h_\circ](t,p) = h_\circ^{-1} \int_t^{t+h_\circ} \int_{U^\lambda} \Lambda(s,\lambda) dm^\lambda(\lambda) ds,$$

in which m^λ is completely determined by p, recall Assumption 2.3.

Note that, for $u \in U^\gamma$, we may have $x + \beta(\cdot, u) + \mathfrak{F}(\cdot; u) \notin \mathbf{X}^{h_\star}_{c_x}$. One needs to approximate φ with the *closest* points in $\mathbf{X}^{h_\star}_{c_x}$. We have the same issue with $\mathbf{P}^{h_\star}_{c_p}$. We define $[\varphi]_{h_\star}$ as an approximation of φ by

$$[\varphi]_{h_\star} = \sum_{(x',p') \in C_{h_\star}(x) \times C_{h_\star}(p)} \omega(x', p' \mid x, p) \varphi(\cdot, x', \cdot, p', \cdot).$$

in which $C_{h_\star}(x)$ (resp. $C_{h_\star}(p)$) denotes the corners of the cube of $\mathbb{R}^d$ (resp. $\mathbb{R}^d$) in which x (resp. p) belongs too and $\omega(\cdot \mid x, p)$ is a weight function.

Moreover, in order to integrate the boundary condition when $l_j \to \ell$ for some $j \in \mathbf{J}$, we define $\overline{\mathbf{L}}^{h_\circ} = \mathbf{L}^{h_\circ} \cup \ell$ and $\mathbf{L}^{h_\circ}_{\mathbf{J}} = \prod_{j=1}^{\kappa} (\partial \mathbf{1}_{\mathbf{J}^c}(j) + \mathbf{L}^{h_\circ} \mathbf{1}_{\mathbf{J}}(j))$. We introduce

$$[\varphi]^\ell(\cdot, c, l, \cdot) = \varphi(\cdot, \mathfrak{C}^\ell_-(c,l), \cdot), \quad (c,l) \in \mathbf{C}^{h_\star}_{\mathbf{J}} \times \overline{\mathbf{L}}^{h_\circ}_{\mathbf{J}}.$$

And finally,

$$[\varphi]^\ell_{h_\star} = [[\varphi]^\ell]_{h_\star}.$$

The discrete counterpart of $\mathcal{L}^{\mathbf{J}}_*$ for all $\mathbf{J} \in \mathcal{P}(\mathbf{K})$ is

$$\begin{aligned} \mathcal{L}^{\mathbf{J}}_h \varphi := \Delta^{h_\circ, t}_i [\varphi]^\ell + \sum_{1 \le i \le d} \mu^i \Delta^{h_\star, x}_i [\varphi]^\ell + \sum_{i \in \mathbf{J}} \Delta^{h_\star, \ell}_i [\varphi]^\ell + \sum_{1 \le i \le d} h_1 \Delta^{h_\star, p}_i [\varphi]^\ell \\ + \overline{\Lambda}[h_\circ] \int_{U^\gamma} \int_{\mathbb{R}^d} \left[\mathcal{I}\left[[\varphi]^\ell_{h_\star}, u \right] (t + h_\circ, \cdot) - \varphi \right] \Upsilon(\gamma, du) dm^\gamma(\gamma). \end{aligned} \tag{6.1}$$

For the sequel, we set $\phi^\circ \in \Phi^{z,\overline{m}}_\kappa$ a control such that $\tau_1^{\phi^\circ} > T$ a.s. and $\phi^a \in \Phi^{z,\overline{m}}_\kappa$ a control such that $\tau_1^{\phi^a} = t$ a.s. and $\tau_2^{\phi^a} > T$ a.s. for $a \in \mathbf{A}$. Thus, the discrete counterpart of $\mathcal{K}$ is

$$\mathcal{K}^h \varphi := \sup_{a \in \mathbf{A}} \mathbb{E}_{\overline{m}} \left[[\varphi]^\ell_{h_\star} (Z^{z,\phi^a}_{t+h_\circ}, P^{t,p}_{t+h_\circ}, M^{z,m,\phi^a}_{t+h_\circ}) \right]. \tag{6.2}$$

We set $\mathring{\mathbf{X}}^{h_\star}_{c_x} := (\mathbf{X}^{h_\star}_{c_x} \backslash \partial \mathbf{X}^{h_\star}_{c_x})$, and $\mathring{\mathbf{P}}^{h_\star}_{c_p} := (\mathbf{P}^{h_\star}_{c_p} \backslash \partial \mathbf{P}^{\circ, h_\star}_{c_p})$.

Our numerical scheme consists in solving, for all $\mathbf{J} \in \mathcal{P}(\mathbf{K})$:

$$0 = \mathbf{1}_{\{\mathbf{J}=\mathbf{K}\}}\left[-\mathcal{L}_h^{\mathbf{J}}\varphi\right] + \mathbf{1}_{\{\mathbf{J}\neq\mathbf{K}\}}\min\left\{-\mathcal{L}_h^{\mathbf{J}}\varphi\,,\ \varphi-\mathcal{K}^h\varphi\right\} \quad \text{on } (\mathbf{T}^{h_\circ}\backslash T)\times\mathring{\mathbf{X}}_{c_x}^{h_\star}\times\mathbf{CL}_{\mathbf{J}}^h\times\mathring{\mathbf{P}}_{c_p}^{h_\star}\times\mathbf{M} \tag{6.3}$$

$$\varphi = g\mathbf{1}_{\{\mathbf{J}=\mathbf{K}\}} + (g\vee\mathcal{K}[g]_{h_\star})\,\mathbf{1}_{\{\mathbf{J}\neq\mathbf{K}\}} \quad \text{on } \{T\}\times\mathring{\mathbf{X}}_{c_x}^{h_\star}\times\mathbf{CL}_{\mathbf{J}}^h\times\mathring{\mathbf{P}}_{c_p}^{h_\star}\times\mathbf{M} \tag{6.4}$$

$$\varphi = g \quad \text{on } \mathbf{T}^{h_\circ}\times\partial\mathbf{X}_c^{h_\star}\times\mathbf{CL}_{\mathbf{J}}^h\times\mathring{\mathbf{P}}_c^{h_\star}\times\mathbf{M} \tag{6.5}$$

Proposition 6.1. *Let* v_h^c *denote the solution of (6.3)-(6.4)-(6.5). Then* $\mathrm{v}_h^c \to \mathrm{v}$ *when* $(h_\star, h_\circ/h_\star) \to 0$ *and* $c \to +\infty$.

Proof. We check that the conditions of [4, Theorem 2.1.] are satisfied as in [2]. □

7 Example: CAT bonds in a *per event* framework for Hurricanes in Florida

Here we focus on a simple example where the controller is an insurance or a reinsurance company which can issue CAT bonds in order to cover its risk in natural disasters.

We will consider CAT bonds of *per event* type. The time-unit will be the year and we fix $\ell = 3$ which corresponds to the average maturity of CAT bonds in years.

We will consider the case of hurricanes occurring on the US Atlantic coast. More specifically, on Florida. The motivation comes from the fact that this region is well exposed, about one hurricane every two years in average, see [10] ; and has an important and increasing insured value about 4000 billion in 2015, see [14].

Thus, we build an example in which an insurer has a strong exposition in Florida against the hurricanes, and can launch CAT bonds to cover it.

We consider a 1-dimension random Poisson measure N, which represents the intensity of arrival and the severity of Hurricanes. We first on the case with a Gamma distribution as a prior.

7.1 Intensity of Hurricanes: the Gamma case

We define the intensity Λ as the function:

$$\Lambda(t,\lambda) = \lambda h(t), \quad (t,\lambda) \in [0,T]\times\mathbb{R}_+^*,$$

in which $h : t \mapsto h(t)$ is a positive continuous function which represents the seasonality of the arrival of hurricanes and some growth according to the global warming. The parameter $\lambda \in U^\lambda := \mathbb{R}_+^*$, which is unknown, represents a level of intensity.

We set $m_0^\lambda = \mathcal{G}(\alpha_0, \beta_0)$ with $(\alpha_0, \beta_0) \in (\mathbb{R}_+^*)^2$ as an initial prior on λ.

Thus, by Example 2.1, we deduce that the process M^{t,m^λ}, starting from $m^\lambda := \Gamma(\alpha_t, \beta_t)$ at $t \in [0,T]$, remains in the family of Gamma distributions and, for all $s \geq t$,

$$M_s = \mathcal{G}\left(\alpha_t + N_s - N_t, \beta_t + \int_t^s h(u)du\right).$$

Moreover, we can define two processes P^α and P^β :

$$\begin{aligned} P^\alpha &= P_t^\alpha + \int_t^\cdot dNs, \\ P^\beta &= P_t^\beta + \int_t^\cdot h(s)ds. \end{aligned}$$

and, by construction, $M = \mathcal{G}(P^\alpha, P^\beta)$.

For the function h, we need to add seasonality. We will add growth's intensity in the Bernoulli case. For the seasonality, especially on big Hurricanes, we refer to [12] in which the authors give a curve based on a kernel density estimation. One close parametric density function over one year can be found in the form:

$$h_0 : [0,1] \to \mathbb{R}_+ \tag{7.1}$$

$$t \mapsto \begin{cases} f_{\hat{\alpha},\hat{\beta}}\left(\frac{t-d_0}{d_1-d_0}\right) & \text{if } t \in (d_0, d_1) \\ 0 & \text{else} \end{cases} \tag{7.2}$$

in which $f_{\hat{\alpha},\hat{\beta}}$ is the density function of the Beta distribution of parameters $(\hat{\alpha}, \hat{\beta}) \in (\mathbb{R}_+^*)^2$.

The Figure 7.1 shows a representation of h_0 close to the one obtained in [12].

7.2 Intensity of Hurricanes: the Bernoulli case

Although the Gamma prior gives parameters that belongs in $\mathbb{R}_+$, in order to remains in the Gamma distribution over time, it requires the form $(t, \lambda) \mapsto \lambda h(t)$ and then the intensity of the whole period is proportional in λ. We introduce a Bernoulli case with three alternatives in which one can place any function depending on time.

With $E : \mathbb{R}_+ \mapsto \mathbb{N}$ the integer part function, we define the intensity as:

$$\Lambda(t, \lambda) = \frac{1}{2}h(t)\left(1 + \frac{E(t)}{T}\lambda\right), \quad (t, \lambda) \in [0,T] \times \{\lambda_1, \lambda_2, \lambda_3\}, \tag{7.3}$$

in which the parameter $\lambda \in \{\lambda_1, \lambda_2, \lambda_3\} \subset \mathbb{R}_+$ represents 3 scenarios of the evolution of the intensity, as a consequence of the global warming.

Following Example 2.2, we can define 3 processes, starting from $p := (p^1, p^2, p^3) \in \mathbb{R}_+^3$ at time $t \in [0,T]$:

$$P^i := p^i - \int_t^\cdot P_s \Lambda(s, \lambda_i)ds + \int_t^\cdot P_{s-}\left[\Lambda(s, \lambda_i) - 1\right] dN_s \tag{7.4}$$

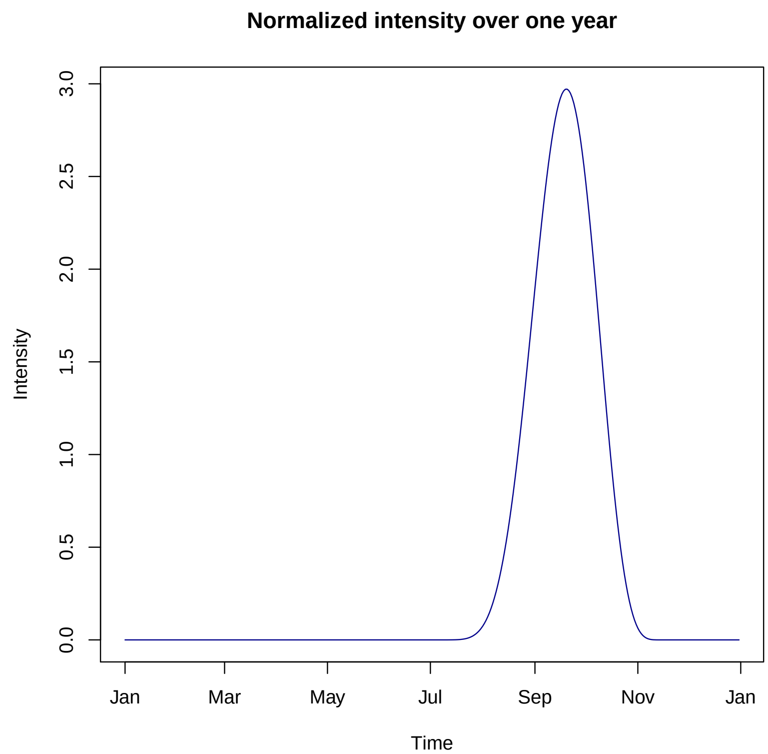

Figure 7.1: Representation of h_0 over one year with $d_0 = 1^{\text{st}}$ July, $d_1 = 15^{\text{th}}$ November, $\hat{\alpha} = 8$ and $\hat{\beta} = 6$.

7.3 Severity of the Hurricanes

As in [10], we use a Generalized Pareto Distribution for the simulation of the severity of the claim, over the exposure of 4000 billion. Their threshold (minimum claim size) is $\mu = 0.25$ billion for an exposure of 2000 billion. Here, we shall use: $\mu = 0.5$, $\sigma = 5$ and $\xi = 0.5$. To fix ideas, the median is 4.5 billion, the quantile at 90% is 22 billion and the quantile at 99.5% is 132 billion. We also bound the distribution by the total exposure of 4000 billion.

Now we define the possible CAT bonds to issue. We will work with *per event* CAT bonds.

We introduce the so-called Occurrence Exceedance Probability (OEP) curve. To this aim, we introduce the random variable:

$$\iota_t := \max_{t \leq s \leq t+1} \int_{\mathbb{R}^*} uN(du, \{s\}),$$

which is the greatest Hurricane in $[t, t+1]$ for $t \in [0, T-1]$. The OEP curve is simply:

$$OEP_{\mathfrak{t}}^t := \inf \left\{ x \in \mathbb{R} : \mathbb{P}\left(\iota_t \leq x\right) \geq 1 - \frac{1}{\mathfrak{t}} \right\}, \quad \mathfrak{t} \geq 0,$$

in which $\mathfrak{t}$ is called the *Return period*. By construction, $OEP_{\mathfrak{t}}^t$ is the quantile of order $1 - 1/\mathfrak{t}$ of ι_t.

The Figure 7.2 shows the corresponding OEP curve with the prior $(p^\alpha, p^\beta) := (25, 50)$.

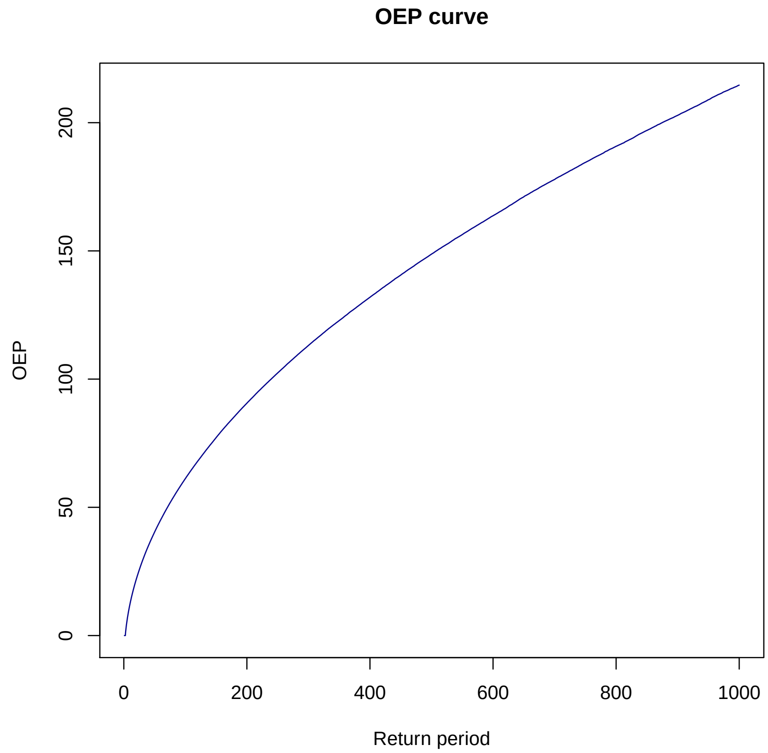

Figure 7.2: Representation of an OEP curve, with the parameter (μ, σ, ξ) defined in the text and with the prior $(p^\alpha, p^\beta) := (25, 50)$.

We now define the set of controls and the output process.

7.4 The set of controls and the output process

Recall that a control ϕ has the form $(\tau_i^\phi, k_i^\phi, n_i^\phi)$. Here n_i is the percentage of the Insured Value in the portfolio of the Insurer and is the notional of the CAT bond. It is fixed to one. We introduce $\{K_1, K_2, K_3, K_4\} := \{10, 50, 200, 1000\}$. We introduce what will be the capacity of the CAT bonds: $\mathfrak{l}_{K_j}^t = OEP_{K_{j+1}}^t - OEP_{K_j}^t$ for $1 \leq j \leq 3$ and $t \in [0, T-1]$.

The value k_i can be chosen in $\{K_1, K_2, K_3\}$ and the associated sets A_{k_i} are defined by:

$$A_{k_i}^t = [OEP_{k_i}^t, +\infty[, \quad i \geq 1.$$

If a Hurricane leads to a cost in $A_{k_i}^t$, then the default of the CAT bond is activated. It remains to define the payout for the insurer in the default case. It corresponds to cover the layer $[OEP_{k_i}^t, OEP_{k_i}^t + \mathfrak{l}_{k_i}^t]$ at a ratio of n_i. We define the payout of the $j-th$ CAT bonds as:

$$F_j(t, x, c, l, n, k, u) := n_j \left[\left(u - OEP_{k_j}^{t-l_j} \right)^+ \wedge \mathfrak{l}_{k_j} \right], \quad j \in \{1, 2\}.$$

Note that, in our example, the risk cannot be covered above the return period of 1000.

We consider the process $X := (X^1, X^2)$ valued in $\mathbb{R}^2$. The first component represents the cash of the Insurer/Reinsurer and the second component represents the risk premium, in term of percentage of the pure premium, of the market about the CAT bonds.

We shall denote by $\rho > 0$ the speed mean return of the price of CAT bonds, by $\rho^\star : \mathbb{R} \mapsto \mathbb{R}$ the increase function of the price after a claim and by $H_0 > 0$ the initial cost of issuing a CAT bond. We set, with $x := (x^1, x^2)$:

$$\begin{aligned} \mu(t,x) &= \begin{pmatrix} \mu + r x^1 \\ -\rho x^2 \end{pmatrix}, \\ \beta(t,x,u) &= \begin{pmatrix} u \\ \rho^\star(u) \end{pmatrix}, \\ H(t,x,a) &= \begin{pmatrix} -H_0 \\ 0 \end{pmatrix}, \\ \overline{C}(t,c) &= \begin{pmatrix} c \\ 0 \end{pmatrix}, \\ F &= \begin{pmatrix} \sum_{j=1}^{\kappa} F_j \\ 0 \end{pmatrix}. \end{aligned}$$

The parameter μ represents the premium rate, the insurer is profitable if $\mu > \mathbb{E}_{\overline{m}}\left[\Lambda(t,\lambda)\right] \int_{\mathbb{R}^*} u \Upsilon(du)$, and $r > 0$ is the constant interest rate.

7.5 Gain function and dimension reduction

The controller wants to maximize, for some $\gamma > 0$, the criteria

$$g(x,c,l,p,m) := -\exp\left[-\gamma \left(x^1 + \frac{H_0}{\ell} \sum_{k=1}^{\kappa} \mathbf{1}_{\{l_k \neq \partial\}} (\ell - l_k) \right)\right] \vee \hat{C}.$$

The right part inside the exponential function compensates the initial cost for remaining CAT bonds, in order to avoid particular behavior of issuing nothing close to the end. We take $\hat{C} := -10^{300}$ which ensures that g is bounded and big enough such that it will not play an essential role.

Note that in the Gamma prior case, we have $P^\beta = P^\beta_t + \int_t^\cdot h_s ds$ which is a function of time. Then, one can avoid it in the numerical scheme since it is a function of time fully characterized by the initial prior.

In the Bernoulli case, one can see that, if we set for the prior

$$p' := \delta p,$$

for some $\delta > 0$, then, for all $s \geq t$, we have $P'_s = \delta P_s$ and then $\mathcal{D}(P'_s) = \mathcal{D}(P_s)$. One can normalized P such that the sum is 1 and avoid the last component.

7.6 The choice of the parameters

We choose here the form and the functions and the parameters for our toy examples. We first describe the Gamma case (for the prior) and then describes the Bernoulli case.

Just after the occurrence of Katrina, the price of the reinsurance was about two or three times greater with a persistence of about two years and can be also seen on the CAT bond market, see Figure 9 in [8]. Thus, we set

$$\rho := 2.$$

Moreover, the estimated return-period of such event is about 20-year return period, see [9]. Since the increase was about two of three times greater, we set

$$\rho^\star(u) := \frac{0.05}{1 - F_{\mu,\sigma,\xi}(u)},$$

in which $F_{\mu,\sigma,\xi}$ denotes the cumulative distribution function of the Pareto distribution of parameters (μ, σ, ξ). Then, here, for a return period of 40 years (recall that we have in average one claim each 2-year period), it gives an increase of 100% of the price.

The insurer has a market share of $e_0 \in]0, 1]$ that we fix at 10%. We shall assume that, the insurer is profitable until $\lambda = 0.65$. Then, the premium rate is

$$\mu := 0.65\, e_0 \int_{\mathbb{R}^*} u\Upsilon(du) = 0.65 \times e_0 \times \left(\mu_0 + \frac{\sigma_0}{1-\xi}\right) = 0.6825.$$

If $k_i = K_j$ with $j \in \{1, 2, 3\}$,

$$r_i = \mathfrak{C}_0(\tau_i, X_{\tau_i}, \alpha_i, \varepsilon_i) = n_i \left[e_0 \left(\frac{1}{K_{j+1}} + \frac{1}{2}\left(\frac{1}{K_j} - \frac{1}{K_{j+1}}\right)\right)\right] \mathfrak{l}_{K_j} \left(1 + x^2 + \varepsilon_i\right). \tag{7.5}$$

Thus, the CAT bond price is decomposed by:

- The part $\frac{1}{K_{j+1}}$ which is the probability that a claim is above the layer within one year and then the payout is the layer
- The part $\frac{1}{K_j} - \frac{1}{K_{j+1}}$ which is the probability that the greatest claim is in the layer, and we multiply it by one half like if it was uniformly distributed in the layer, which is greater than the true value.
- The factor x_2 is the risk aversion of the market, and ε_i is some random value about the price the coupon.

Finally, the cost of issuing a CAT bond is fixed at: $H_0 := 0.0025$, the interest rate is fixed at $r := 1\%$ and the market share at $e_0 := 10\%$.

Remark 7.1. *In these examples, we deal with per event CAT bonds. One also can deal with aggregated losses within the period. In this case, one needs to remember the current accumulation of claims and to introduce another dimension in the output process X.*

Remark 7.2. *In practice, in general, a partial default below 70%-80% of the capacity does not end the CAT bonds: the coupon is reduced by the proportional loss and another loss may lead to the complete default, using the same limits. Here, for simplification, the CAT bond ends whenever the layer is attained.*

Remark 7.3. *Note that the function* $\Psi(x, c, l, p) := \frac{\mu}{r} + x^1 + \delta$ *satisfies the conditions of Proposition 5.1, for* $\delta > 0$ *great enough.*

Note that, in this example, we did not add any global warming effect, it will be added in the Bernoulli case. Actually here, we could only add a deterministic global warming effect since the Bayes stability requires an intensity of the form $\Lambda(t, \lambda) = \lambda h(t)$.

7.6.1 With a convex hull of Dirac masses

In this case, the intensity grows over time, recall (7.3). We fix $\lambda_1 = 0.2, \lambda_2 = 0.3, \lambda_3 = 0.4$ and $P_0^1 = P_0^2 = P_0^3 = \frac{1}{3}$, recall (7.4).

To be consistent, we say that the premium rate also rises over time following the rise of intensity, but by 35%, and then is:

$$\mu(t, x) = \begin{pmatrix} \mu\left(1 + 0.35\frac{t}{T}\right) + rx^1 \\ -\rho x^2 \end{pmatrix}, \quad (t, x) \in [0, T] \times \mathbb{R}^2.$$

We assume that the market is updating the OEP with:

$$OEP^t := OEP^0 \left(1 + 0.35\frac{t}{T}\right), \quad t \in [0, T].$$

7.7 Results

Recall that, for each CAT bond that the insurer can issue, we need to add its characteristics and then the complexity increases hugely in κ, depending on possible policies. Thus, in our simulation, we use $\kappa = 2$ and thus, the controller can choose at most 2 layers among the three available (recall them in term of return periods: [10, 50], [50, 200] and [200, 1000] which correspond to [1.23, 4], [4, 9], and [9, 21.5] in billion dollars).

7.7.1 With the Gamma prior

In Figure 7.3, we provide a simulated path of the optimal strategy in which the Pareto distribution is discretized in 2500 points (the highest possible value is 49 billion dollars). The top left graphic describes the control played by the insurer. The top part represents the issue of CAT bonds, the level is the lower bound of the layer. The bottom part represents the running CAT bonds with respect to the layer. The double dash says that two CAT bonds at the same layer are running. The top right graphic describes the arrival of natural disasters. The bottom part gives the size of the claim of the insurer while the top part gives the payoff of the CAT bond(s). The middle left graphic describes the evolution of the cash of the insurer. The middle right graphic gives the evolution oh X^2, the price penalty of the CAT bonds which appears in (7.5). The bottom left graphic gives the evolution of the mean of the estimated distribution of λ_0, defined by $\frac{P^\alpha}{P^\beta}$, and the bottom right graphic gives the evolution of the standard deviation, defined by $\frac{\sqrt{P^\alpha}}{P^\beta}$.

At the beginning, the insurer does not issue any CAT bond. Since we start in January, there is no risk to experiment a claim and thus the insurer delay the issue. Just when the season starts, he first chooses to issue two CAT bonds on the layer $[200, 1000]$. Recall that it is the highest layer which corresponds to $[9, 21.5]$ in billion dollars. It is possible to have a claim highly above the layer and having a double cover on this big layer gives, indirectly, a cover against huge claims above the layer (recall that the maximum claim size is 49 billion dollars). He renews each CAT bond at the maturity until he meets a claim with a return period above 1000 during the 5^{th} year. He gets the associated payoff. Despite the huge increase of the price of CAT bonds, by almost 400%, he immediately issues a new one on the layer [200, 1000], but only one. He waits the next season, with a better expected price, to issue the other one. After, he follows this strategy to the end, except very close to the end where he optimizes the cost of CAT bonds.

In Figure 7.4, we represent the approximated density (by kernel estimation) of the total cash of the insurer at the end of the 30 years. On the left, it is the case with $\lambda_0 = 0.6$ (as it is also the case in Figure 7.3) and on the right with $\lambda_0 = 0.5$, i.e. what believes the insurer at the beginning. The solid curve is the case when the insurer plays the optimal control and the dashed curve is when he never issues any CAT bond. We also add the quantiles at 99.5% in term of losses, see the legend. In the case with $\lambda_0 = 0.6$ (left), from which the paths in Figure 7.3 come from, we can see that the standard deviation is reduced. And the quantile at 99.5% is strongly reduced. One can observe that the case $\lambda_0 = 0.6$ strongly reduces the expected net return in average.

We now look at the case with a discretization of 500 of the Pareto distribution. In particular, the maximum claim size is 21.4 billion which does not exceed the maximum layer $[9.0, 21.5]$. In general, the risk is lower. In Figure 7.5, we show a simulated path. This time, the insurer chooses to get two CAT bonds at the layer $[50, 200]$. Actually, with this discretization, the layer $[200, 1000]$ appears to be less competitive since the discretization of 500 leads to a lower expected payoff. In the first years, the expected intensity is revised higher and the relative price of the layer $[10, 50]$ decreases (this layer requires the highest coupon since it is frequently hit). At the 4^{th} year, he changes his strategy and gets one CAT bond on the layer $[10, 50]$ and the other one on the layer $[50, 200]$. A catastrophe above the return period of 200 occurs at the 20^{th} year and both CAT bonds end. He prefers to wait the next season because of the consecutive price increase. Note that, in the previous cases (with Pareto distribution discretized in 2500 points), he was never without any CAT bond, even after an increase of 400%. Then, he continues his strategy to get a CAT bond on the layer $[10, 50]$ and the other one on the layer $[50, 200]$, until the end.

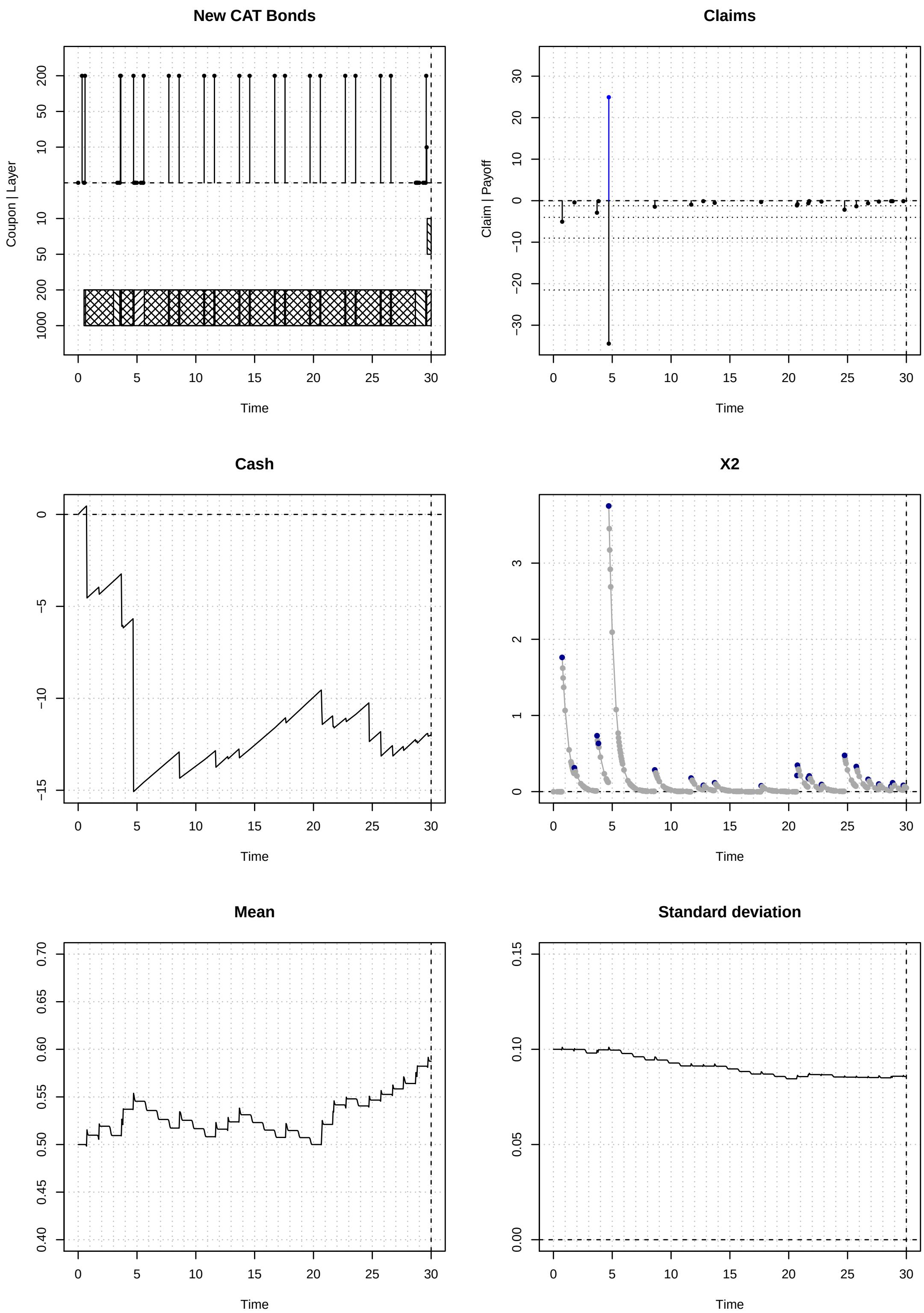

Figure 7.3: Simulated path of the optimal strategy of the insurer.

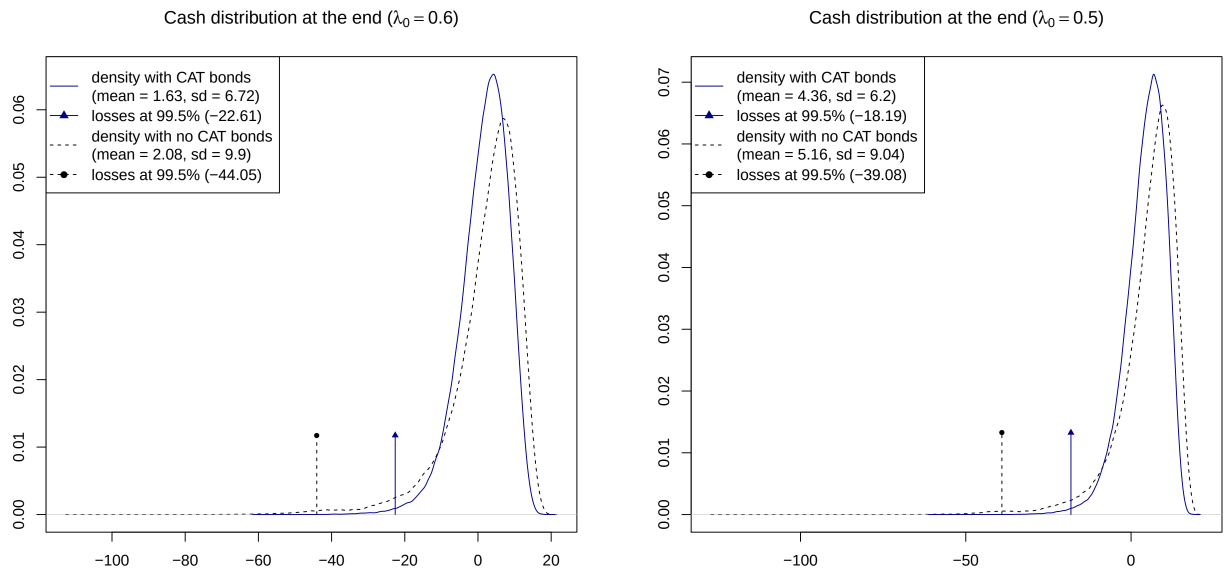

Figure 7.4: Cash distribution (with 200 000 simulations) for $\lambda_0 = 0.6$ (left) and $\lambda_0 = 0.5$ (right) with the optimal control (solid dark blue) and without any CAT bond (dashed black).

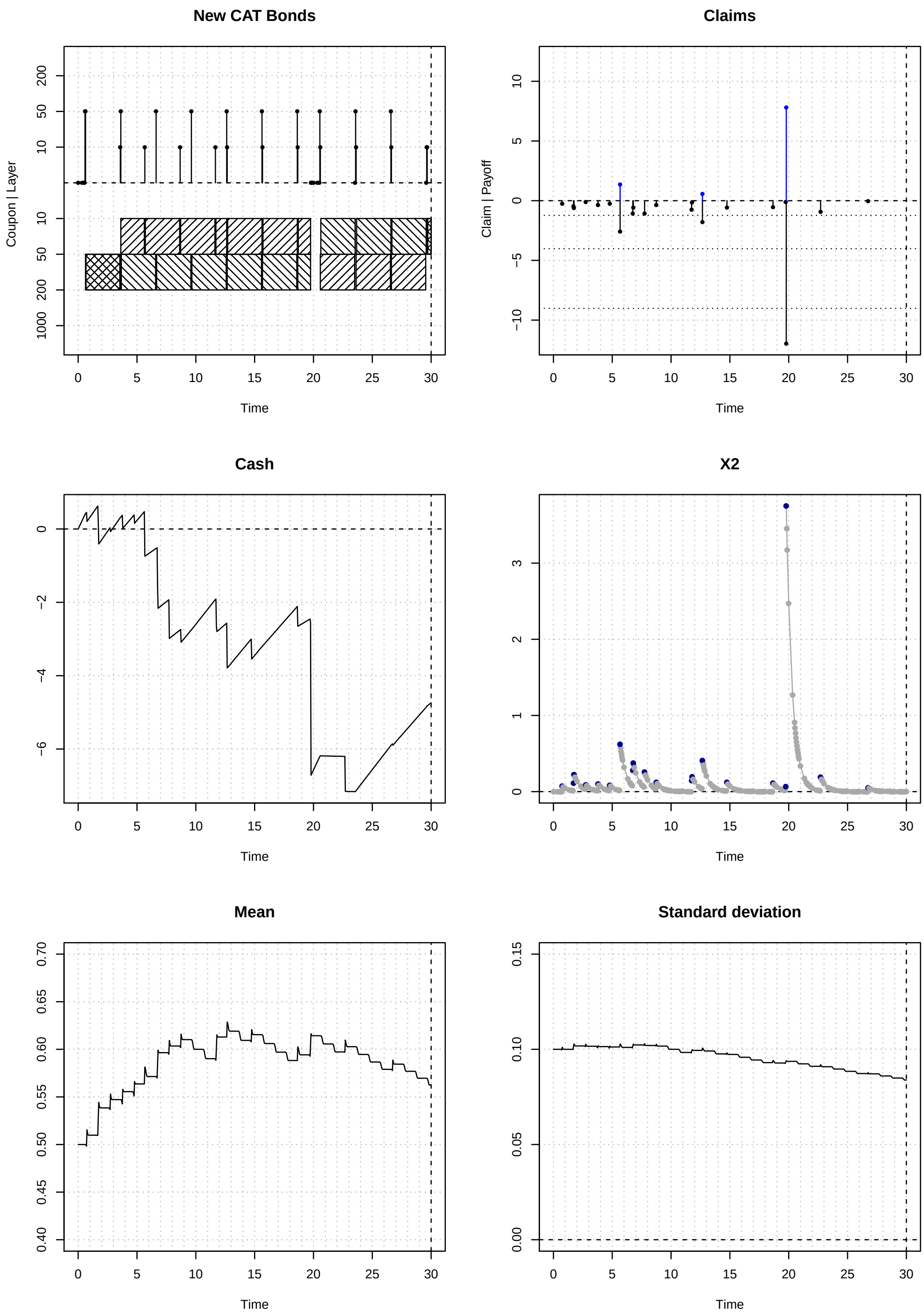

Figure 7.5: Simulated path of the optimal strategy of the insurer.

7.7.2 With the Bernoulli prior

In Figure 7.6, we provide a simulated path of the optimal strategy in which the Pareto distribution is discretized in 2500 points (recall that the highest possible value is 49 billion dollars). As in the Gamma prior case, the insurer chooses to get two CAT bonds at the higher layer. When he experiences a huge claim during the second year, he still gets twice the layer but prefers to wait before to take a new CAT bond, according to the huge rise of the price. He waits the next year and restarts the same strategy until the 12^{th} year. Then, he issues CAT bonds on the layer $[50, 200]$ and $[200, 1000]$ until close the end.

The estimated probabilities on λ_0 evolve slowly at the beginning since λ_0 has an impact which rises over time.

In Figure 7.7, we represent the approximated density (by kernel estimation) of the total cash of the insurer at the end of the 30 years. On the left, it is the case with $\lambda_0 = 0.4$ (as it is also the case in Figure 7.3) and on the right with $\lambda_0 = 0.3$. The legend is the same as in Figure 7.4 and we get close distributions.

We now look at the case with a discretization of 500 of the Pareto distribution and show a simulated path in Figure 7.8. As in the Gamma prior case, at the beginning, the insurer chooses to get two CAT bonds at the layer $[50, 200]$. He follows this strategy until he meets a huge claim in the 16^{th} year. He waits the next season and restarts the same strategy. At the 24^{th} year, he chooses to issue CAT bonds on two different layers, at $[50, 200]$ and $[10, 50]$. As in Figure 7.5, this results in a change on the belief on the intensity.

Finally, in Figure 7.9, we display the distribution of the probabilities on λ_0. This highlights the fact that it is very difficult to estimate it with observations through time.

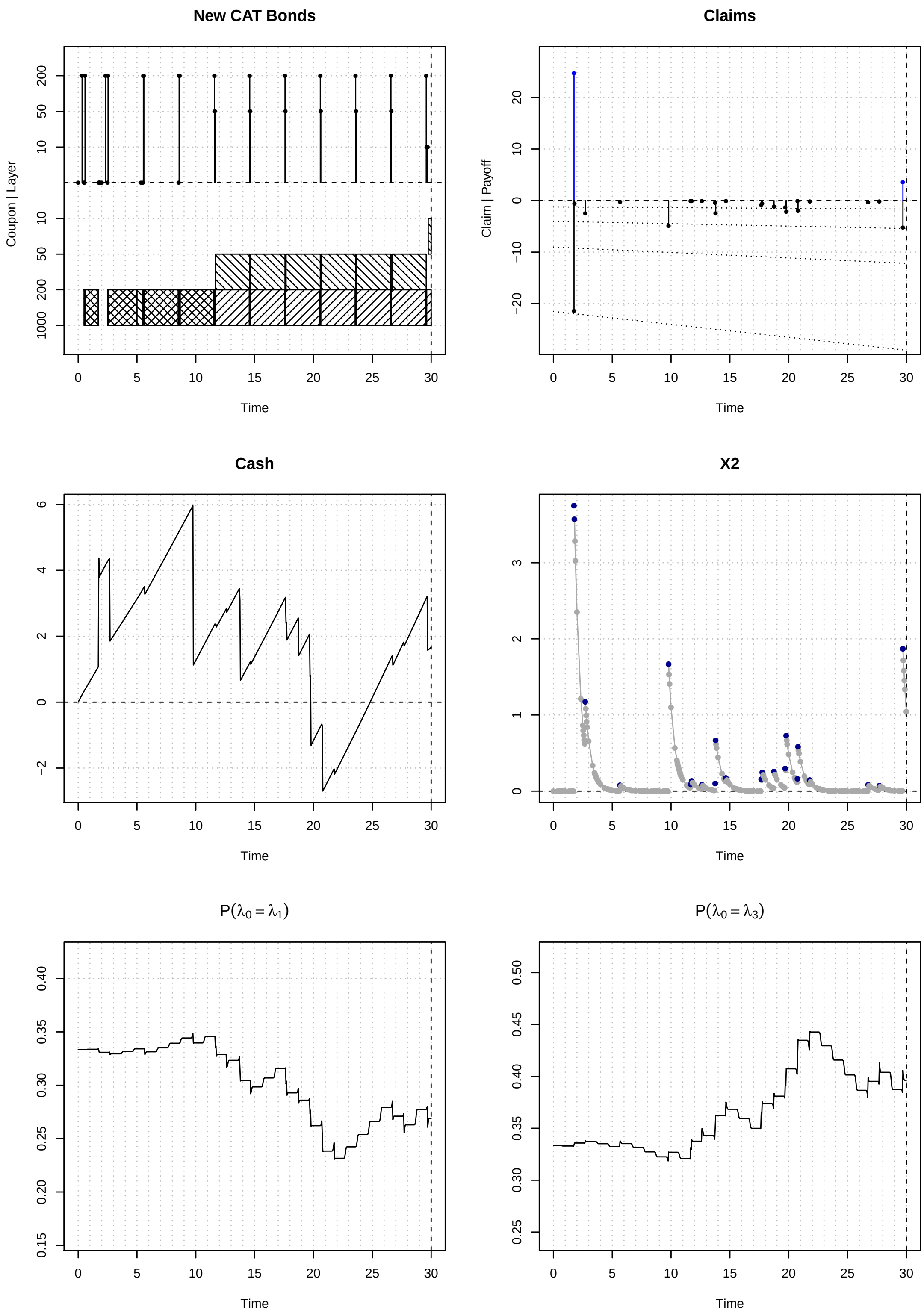

Figure 7.6: Simulated path of the optimal strategy of the insurer.

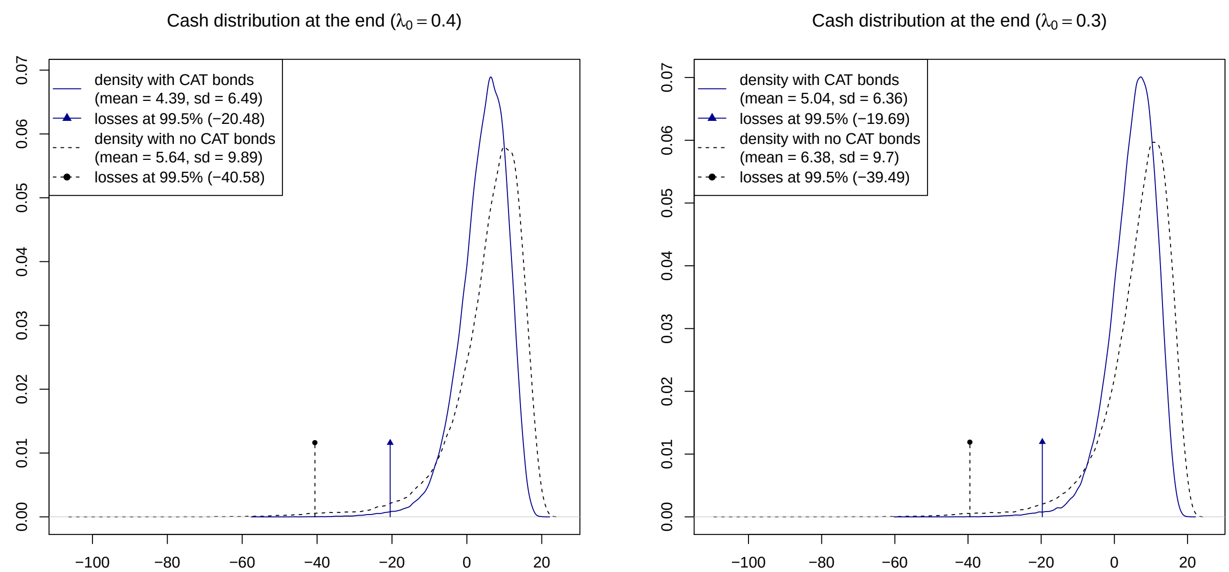

Figure 7.7: Cash distribution (with 200 000 simulations) for the increase parameter $\lambda_0 = 0.4$ (left) and $\lambda_0 = 0.3$ (right) with the optimal control (solid dark blue) and without any CAT bonds (dashed black).

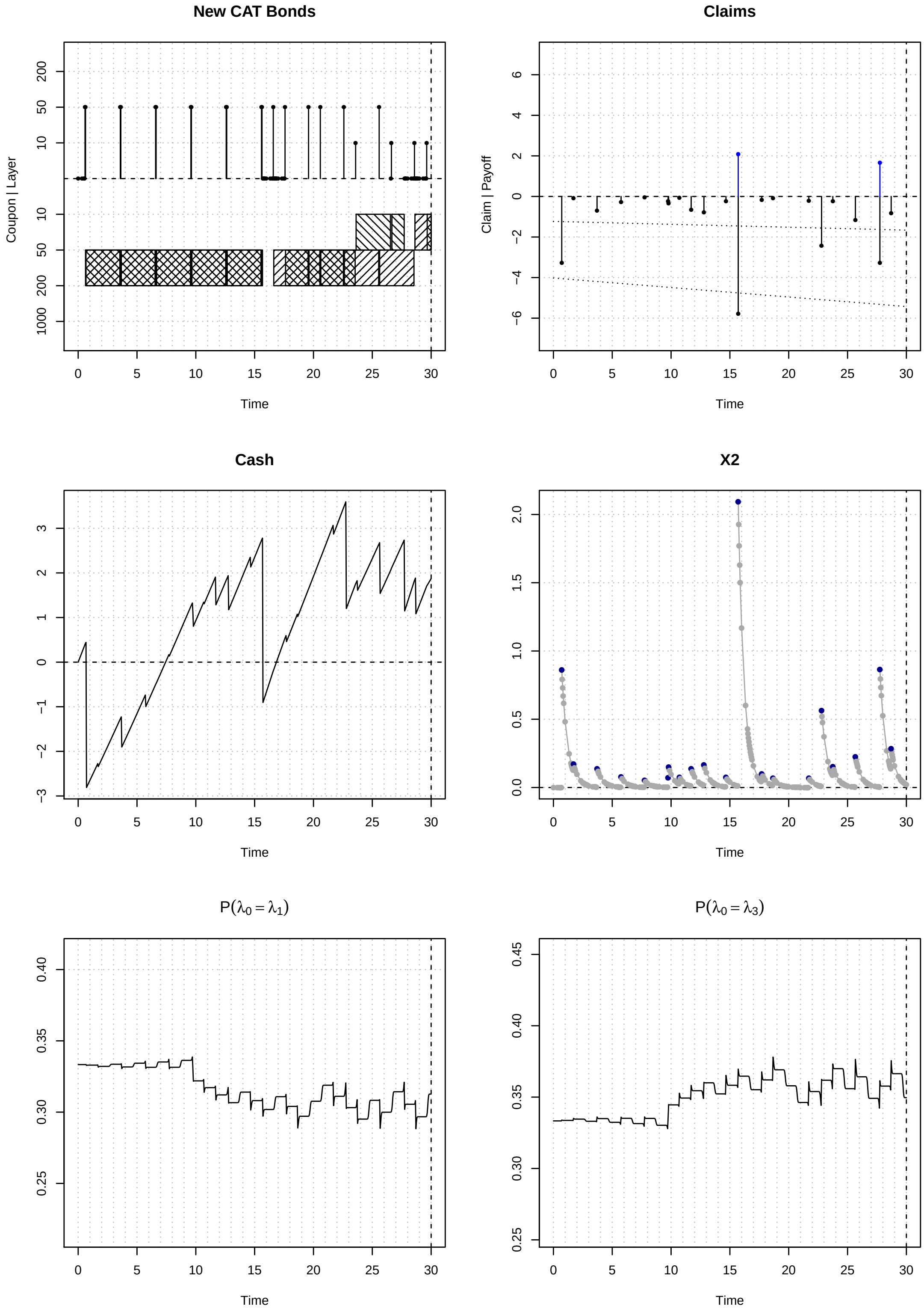

Figure 7.8: Simulated path of the optimal strategy of the insurer.

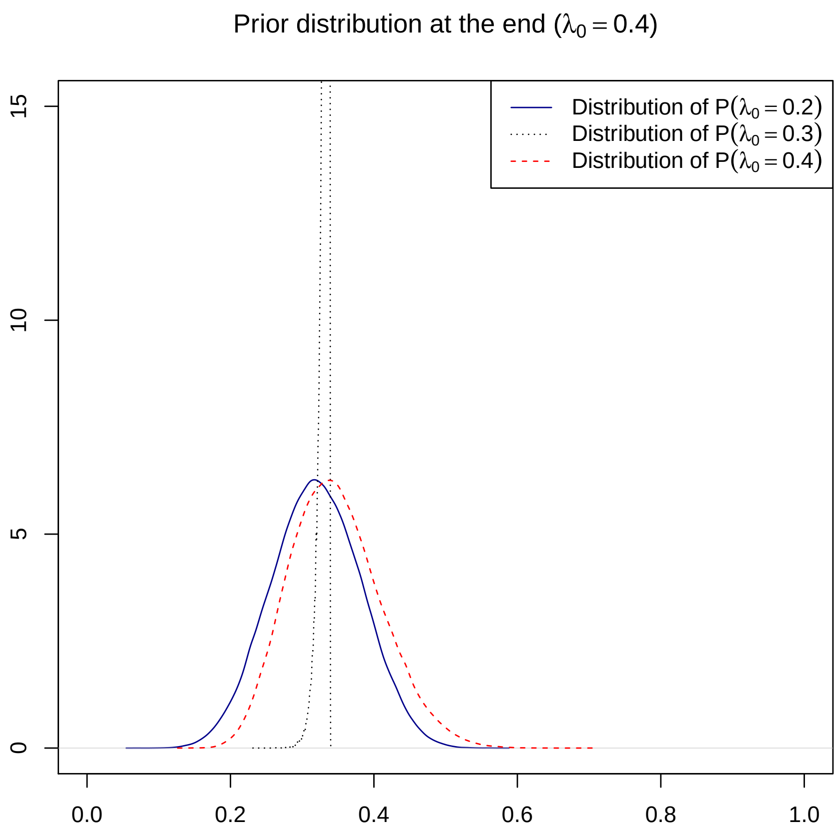

Figure 7.9: Distribution of the probabilities on λ_0 at the end (with 200 000 simulations).

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